MODAL FRAME INCOMPLETENESS. AN ACCOUNT THROUGH SECOND ORDER LOGIC[1]

MIRCEA DUMITRU

Abstract. Propositional modal logic is usually viewed as a generalization and extension of propositional classical logic. The main argument of this paper is that a good case can be made that modal logic should be construed as a restricted form of second order classical logic. The paper makes use of the embedding of modal logic in second order logic and henceforth it goes on examining one aspect of this second order connection having to do with an incompleteness phenomenon. The leading concept is that modal incompleteness is to be explained as a kind of exemplification of standard order incompleteness. Moreover the modal incompleteness phenomenon is essentially rooted in the weaker expressive power of the language of sentential modal logic as compared to the stronger expressive power of the language of second order logic.

Keywords: Henkin-semantics; language of second-order logic; modal incompleteness; standard semantics; System VB.

1. SOME BASIC NOTIONS OF MODAL METALOGIC; FRAMES AND VALIDITY

In the beginning I shall sketch the metalogic framework of my research, following [9], [10], [11], [12], [16]. People acquainted with what follows in this section are advised to move on to the next section. A general interpretation for a language of sentential modal logic (LSML) has three component parts: a set of worlds W, an accessibility relation R, and for each world $w \in W$, an associated assignment of truth-values to sentence-letters. There is also a particular world designated the actual world; but in the investigation that follows the latter component plays no role.

It is useful to redefine our notion of general interpretation in two stages. A *Kripke-frame*[2] is a pair $F = (W, R)$, where W is a set of worlds, and R is a binary

[1] Previously published in Melvin Fitting (ed.), *Selected Topics from Contemporary Logics. Landscapes in Logic 2*, College Publications, 2021, pp. 183–202.

[2] Whenever ambiguity does not rear its ugly face, I shall suppress the qualifying phrase "Kripke" when I speak about frames or models. However, when the contrast between Kripke-frames or models, on the one hand, and general-frames or models, on the other hand, is crucial for my explanation and for understanding the issue, I will restore the phrase which marks the contrast.

relation on W. A frame F is said to be reflexive (symmetric, transitive, etc.) if and only if (iff) R_F is reflexive (symmetric, transitive, and so on). A *Kripke model* for LSML is a pair $M = (F, V)$, where F is a frame and V is a function defined for each sentence-letter π of LSML. V assigns each such π a subset of W (intuitively, the worlds at which π is true); thus $V(\pi) \in \mathcal{P}(W)$. If $M = (F, V)$ and $F = (W, R)$, we may also write $M = (W, R, V)$ or $M = (W_M, R_M, V_M)$. If $M = (F, V)$, then M is said to be based on F.

We now define three semantic concepts, that of (i) a formula's being true at a world in a model M, (ii) of a formula's being valid in a model M, and (iii) of a formula's being valid in a frame F. To define being true at a world in a model M, we recursively define a relation $\vDash$ (read: "verifies"), a subset of $(M, w) \times \mathrm{Prop}(L)$, as the least relation satisfying:

EA: $(M, w) \vDash \pi$ iff $w \in V_M(\pi)$, for each sentence-letter π in LSML,

E¬: $(M, w) \vDash \neg\Phi$ iff $(M, w) \nvDash \Phi$,

E&: $(M, w) \vDash \Phi \,\&\, \Psi$ iff $(M, w) \vDash \Phi$ and $(M, w) \vDash \Psi$,

E∨: $(M, w) \vDash \Phi \vee \Psi$ iff $(M, w) \vDash \Phi$ or $(M, w) \vDash \Psi$,

E⟶: $(M, w) \vDash \Phi \longrightarrow \Psi$ iff $(M, w) \nvDash \Phi$ or $(M, w) \vDash \Psi$,

E□: $(M, w) \vDash \Box\Phi$ iff $\forall u \in W_M$ (If $R_M(w, u)$ then $(M, u) \vDash \Phi$),

E◇: $(M, w) \vDash \Diamond\Phi$ iff $\exists u \in W_M(R_M(w, u)$ and $(M, u) \vDash \Phi)$.

We can define now our two notions of validity: A formula Φ is *valid in a model* $M = (W_M, R_M, V_M)$ iff $(M, w) \vDash \Phi$ for every $w \in W_M$; we may write this as $\vDash_M \Phi$. A formula Φ is valid in a frame $F = (W_M, R_M)$ iff for every model M based on F, the formula Φ is valid in M; we may write this as $\vDash_F \Phi$.

These definitions concern only single formula validity. However, we can define complementary notions of semantic consequence in the same way: $\Sigma \vDash_M \sigma$ iff for every $w \in W_M$, if every member of Σ holds at w then σ holds at w; in other words, if $(M, w) \vDash \Sigma$ then $(M, w) \vDash \sigma$. And $\Sigma \vDash_F \sigma$ iff for every M based on F, we have $\Sigma \vDash_M \sigma$. (I shall assume Σ finite, so that $\Sigma \vDash_M \sigma$ iff $\vDash_M \bigwedge \Sigma \longrightarrow \sigma$, where $\bigwedge \Sigma$ is the conjunction of all members of Σ. The well-known modal systems S5, S4, B, T all have the finite semantic consequence property: if Σ is infinite and $\Sigma \vDash_F \sigma$, then there is a finite subset Σ_0 of Σ such that $\Sigma_0 \vDash_F \sigma$; so the restriction to finite Σ is not significant for these systems.)

We define now the main metalogical concepts of interest for this paper. A deductive system of modal logic is either the system K[3] or a proper extension of K obtained by adding a decidable collection of axiom-sequents to K at least one of which is not itself K-derivable. A deductive system S is *sound with respect to* a class of frames $\mathcal{K}$ iff: if $\Sigma \vdash_S \sigma$, then for every frame $F \in \mathcal{K}$, we have $\Sigma \vDash_F \sigma$. If $\Sigma \vDash_F \sigma$ for every

[3] The system K, so-called in honor of Saul Kripke, one of the inventors of possible worlds semantics, is the simplest modal system. We get it axiomatically by adding the axiom-sequent K: $\Box\,(\Phi \longrightarrow \Psi) \longrightarrow (\Box\,\Phi \longrightarrow \Box\,\Psi)$ and the rule of necessitation Nec: if $\vdash_K \Phi$ then $\vdash_K \Box\,\Phi$, to any sound and complete deductive system of non-modal sentential logic.

frame $F \in \mathcal{K}$, then the sequent $\Sigma \vdash_S \sigma$ is said to be $\mathcal{K}$-valid. So S is sound with respect to $\mathcal{K}$ iff every S-provable sequent is $\mathcal{K}$-valid. A deductive system S is *complete with respect to* a class of frames $\mathcal{K}$ iff: if $\Sigma \vDash_F \sigma$ for every frame $F \in \mathcal{K}$, then $\Sigma \vdash_S \sigma$. Equivalently, S is complete with respect to $\mathcal{K}$ iff every $\mathcal{K}$-valid sequent is S-provable. A deductive system S is *characterized* by a class of frames $\mathcal{K}$ iff S is both sound and complete with respect to $\mathcal{K}$, i.e. the S-provable sequents and the $\mathcal{K}$-valid sequents are the same. Lastly, a deductive system S is *complete simpliciter* iff there is some class $\mathcal{K}$ of frames such that S is characterized by $\mathcal{K}$.

Since its inception, at the end of 1950s and the beginning of 1960s, the possible worlds semantics has become an enormously successful program. Due to this powerful and flexible formal tool many modal systems, which by then had been investigated only with axiomatic means, got a real and insightful semantic interpretation. The methodological success of characterizing modal systems motivated the reasonable hypothesis that every modal deductive system is complete in the absolute sense defined above, i.e. it is characterizable by a class of Kripke-frames.

Today we know that this hypothesis is false, and we owe this piece of knowledge to the research of some modal logicians, like Kit Fine [8] (see also [11]), S. Thomason [18], Johan Van Benthem [1, 2, 3] or G. Boolos and G. Sambin [5], Cresswell [6], who are some of the important names in this field. And my aim here is just to show, first, what a semantic incomplete system looks like, and then to look for an explanation of this interesting and also curious semantic phenomenon.

One way of putting this fact of there existing *incomplete* propositional modal logics is to say that there is a class of frames $\mathcal{F}$ that characterize a logic L that is not axiomatizable. A similar phenomenon occurs in second order classical logic, where one quantifies over subsets of the domain as well as over individuals. Classical second order validity is not axiomatizable; it too displays incompleteness aspects. To that problem Henkin offered a solution, which he called *general models*. In these, set quantifiers are restricted to a designated collection of subsets of the domain, and do not range over all subsets. Validity with respect to general models is axiomatizable. Henkin's general models were, in fact, the inspiration for the introduction of general frames into modal semantics. Against the background sketched above, the gist of this paper is to give a second-order-based-explanation of modal incompleteness. The technical apparatus which is deployed in my argumentation builds upon a case of the embedding of modal logic in second order logic. The leading concept is that modal incompleteness is to be explained in terms of the incompleteness of standard second order logic, since modal language is basically a second order language. That is, the paper shows that modal frame incompleteness is a kind of exemplification of classic second order incompleteness. Roughly, all this goes as follows. A completeness proof for an axiomatization or for a natural deduction system of a modal logic can be formalized in second order logic with standard semantics. At a certain point in the formalized proof, we need the existence of a certain set of possible worlds. Of course that set is in the range of second order quantifiers in standard second order models, but might not be in the allowed quantifier range of some general (Henkin) second order models. Thus the argument can be carried out in standard second order logic, which is not axiomatizable, and cannot always be carried out in the axiomatizable logic corresponding to general second order models. In effect, modal frame incompleteness is seen as an exemplification of classical second order incompleteness, having strong family resemblance with it. Consequently, what follows falls into three sections. In the next section I shall present a very simple incomplete system, which was discovered by Johan Van Benthem. Then, in the following sections, I shall sketch two semantic systems for the language of second-order logic, which are needed in the last section for building–in its essentials–an explanation which ties this semantic phenomenon with the more profound fact that every second-order deductive system which is sound with respect to the standard semantics for its language is bound to be incomplete with respect to that semantics. Beyond the clarification of certain technical aspects, the net result of my approach is that it sheds light on some unexpected connections between important results, which prima facie seem to be unconnected. To my mind, such links are very instrumental in pushing forward our subject, which as far as logic is concerned, let's remember what Frege said, is nothing more and nothing less than the truth itself.

2. THE INCOMPLETE SYSTEM VB

We shall show following [1], [2] and [10] that a certain system of modal logic, VB (to honor Van Benthem) is incomplete, i.e. it is a system which is characterized by no class of frames $\mathcal{K}$. So a better tag for such a system would be "uncharacterizable system". The real form of uncharacterizability results is that of a conditional: "if system S is sound with respect to $\mathcal{K}$ then S is not complete with respect to $\mathcal{K}$".

We first define the system K* to be the system K plus an additional axiom-sequent, viz.

K: $\square(A \longrightarrow B) \longrightarrow (\square A \longrightarrow \square B)$ plus the axiom sequent $\Diamond \square A \vee \square A$.

I state without proof the following result which is a Sahlqvist case, [15]:

Claim 2.1: K* *is characterized by the class of frames in which every world is either a dead end or else is one step removed from a dead end;* w *is a dead end if it can see no world:* $\forall u \neg Rwu$*;* w *is one step removed from a dead end iff it can see some world which is a dead end:* $\exists v(Rwv \,\&\, \forall u \neg Rvu)$.

We define now the system VB to be the system K plus the axiom-sequent $\Diamond \square A \vee \square(\square(\square B \longrightarrow B) \longrightarrow B)$. The proof that VB is incomplete proceeds in two steps. Here, and in the following, I draw essentially on [10]. "First we show *Step 1*: Every frame for VB is a frame for K*. (A frame F is said to be a frame for a system S if whenever $\Sigma \vdash_S \sigma$ we have $\Sigma \vDash_F \sigma$.) Then we show *Step 2*: $\Diamond \square A \vee \square A$ is not a theorem of VB. We have to make clear why is this establishing the incompleteness of VB. The reason is as follows: Suppose $\mathcal{K}$ is a class of frames with respect to which VB is sound. Then, once we have established Step 1, we may conclude that there are no counter examples to K*-sequents based on frames in $\mathcal{K}$. In that case, $\vDash_{\mathrm{K}} \Diamond \square A \vee \square A$. Hence, there is a sequent valid in $\mathcal{K}$ which, granted Step 2, is not derivable in VB. So VB is incomplete with respect to $\mathcal{K}$

(the class of frames with respect to which it is sound). I give now a few details and hints concerning the proof of these two steps. Proof of Step 1 requires a single lemma. Proof of Step 2 requires a sequence of lemmas."

Lemma 2.2: *Every frame for* VB *is a frame for* K*.

Proof. Suppose $F = (W, R)$ is not a frame for K*. We will be able to show then that F is not a frame for VB either. Since K* = K + $\Diamond\Box A \vee \Box A$, and every frame *is* a frame for K, it follows that $\nvDash_F \Diamond\Box A \vee \Box A$. As one knows this means in turn that there is a model M based on F and a world $w \in W_M$, such that $(M, w) \nvDash_F \Diamond\Box A \vee \Box A$. If w were a dead end or could see a dead end, $\Diamond\Box A \vee \Box A$ would hold at w. Hence w is neither a dead end nor can it see one. Let u be a world that w can see. For the language $\{A, B\}$ we define a new model M' based on F in the following way: $V(A) = \emptyset$, i.e., A is false at every world. $V(B) = W - \{u\}$, where u is the previously chosen world which w can see. The lemma is established by showing that $(M', w) \nvDash \Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B)$. Clearly, $(M', w) \nvDash \Diamond\Box A$: though w can see some worlds, none of them are dead ends, and $V(A) = \emptyset$ (so at any world v that w can see, $\Box A$ fails). To show that $(M', w) \nvDash \Box(\Box(\Box B \longrightarrow B) \longrightarrow B)$ we argue that $(M', w) \vDash \Diamond(\Box(\Box B \longrightarrow B)\&\neg B)$. This follows from $(M', u) \vDash \Box(\Box B \longrightarrow B)\&\neg B$ since w can see u. For this latter claim, observe that since $V(B) = W - \{u\}$, $(M', u) \vDash \neg B$. Also, u is not a dead end. Let v be any world that u can see. If $v = u$, then since $u \notin V(B)$, $(M', v) \vDash \Box B \longrightarrow B$. (The conditional has a false antecedent.) And if $v \neq u$, since $v \in V(B)$, so again $(M', v) \vDash \Box B \longrightarrow B$. ($v$ makes the consequent B true.) Hence every world u can see makes $\Box B \longrightarrow B$ true which gives us the required result that $(M', u) \vDash \Box(\Box B \longrightarrow B)$ and the Lemma follows, since we have shown that if $\nvDash_F \Diamond\Box A \vee \Box A$, then $\nvDash_F \Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B)$. ■

It remains to establish Step 2, that $\Diamond\Box A \vee \Box A$ is not a theorem of VB. The obvious way to try to do this is to look for a frame *for* VB in which $\Diamond\Box A \vee \Box A$ is not valid. But according to Lemma 1, there are no such frames. A less obvious procedure is to characterize a class D of general models about which we can prove that (a) all the VB-derivable sequents are valid in (every M in) D, but (b) $\Diamond\Box A \vee \Box A$ fails in at least one M in D.

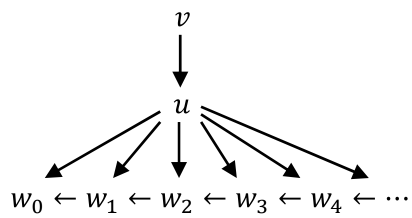

Figure 1: Recession frame.

Remark 2.3: This suffices because to show that a sequent X is not provable in a system S, we only need to find some property or other which is possessed by all S-provable sequents but not by X. The class of general models D that we look for here is a subclass of the class of general models definable on a certain frame F, which is an example of a recession frame. F is defined as follows (see Figure 1.):

• $W = \{v, u, w_0, w_1, w_2, \dots\}$; that is, an infinite collection of worlds $\{w_i\}_{i \in \mathbb{N}}$ ($\mathbb{N}$ is the set of all natural numbers), and two additional worlds v and u.

• w_j can see w_i for every $i < j$; u can see every w_i; v can see u.

Next, we define the set G of subsets of W as the set of all finite subsets of $W - \{u\}$ and their complements in W:

• $G = \{X \subset W;\ X$ is finite and $u \not\in X$ or $\overline{X}$ is finite and $u \not\in \overline{X}\}$. (Note that when $u \in X$, then X is infinite; the set G is closed under complements.)

More intuitively, the set G, which may be called the set of allowable propositions in W, can be generated by the following procedure:

> Take each finite subset X of W or infinite subset Y of W which has a finite complement in W;
>
> If X is finite and $u \not\in X$, then put X in G;
>
> If Y is infinite and has a finite complement in W and $u \in Y$, then put Y in G;
>
> Nothing else will qualify as an element of G.

So, G is a set of sets each of which element being either finite, provided u is not a member of it, or infinite, provided it is the complement of a finite set and it (the infinite set) has u as one of its members. The class D of general models that we are interested in is the class of general models based on F with the valuation function V defined as:

• V is such that $V(\pi) \in G$ for every sentence-letter π.

Lemma 2.4: *Let M be a model (F, V), where F is the general recession frame defined above and V is the valuation function restricted as indicated above. If σ is a sentence whose sentence-letters are $\pi_1, .., \pi_n$, and $V(\pi_i) \in G$ where $1 \leq i \leq n$, then $\{w \in W;\ (M, w) \vDash \sigma\} \in G$.*

Proof. Call the set of worlds at which a sentence is true the sentence's worldset. Then the Lemma says that the property of a sentence of having a worldset in G is preserved by the various ways of forming more complex from less complex sentences. The proof is by induction on the length of modal formulae. It suffices to show that $\neg, \vee, \Box$ preserve the property indicated in the lemma above, since the three operators form an expressively complete set. Let's observe that the Lemma 2.4 holds since the set G has the following properties:

(a) If $A \in G$, then $\overline{A} \in G$,

(b) If $A \in G$ and $B \in G$, then $A \cup B \in G$,

(c) If $A \in G$, then $\{w \in W;\ \forall w' \in W(Rww' \Rightarrow w' \in A)\} \in G$. ■

The role that this result plays in the overall strategy of the proof of VB's incompleteness is more clearly brought about by the following corollary:

Corollary 2.5: *In every model in D, the worldset of any sentence is an element of G.*

We can show now that every VB-sequent is D-valid, i.e. valid in every model $M \in D$.

Lemma 2.6: *If $\Sigma \vdash_{\mathrm{VB}} \sigma$ then $\Sigma \vDash_M \sigma$ for every $M \in D$.*

Proof. Since every K-provable sequent is valid in the class of any model, we only have to show that for every $M \in D$, validity in M is preserved by the rule of theorem introduction (**TI**) using a substitution instance of the characteristic axiom-sequent of VB, say $\vdash \Diamond\Box p \vee \Box(\Box(\Box q \longrightarrow q) \longrightarrow q)$. We argue that in any model M in D, $(M, w) \vDash \Diamond\Box p \vee \Box(\Box(\Box q \longrightarrow q) \longrightarrow q)$ for every $w \in W_M$. Since w_0 is a dead end, $(M, w_0) \vDash \Box(\Box(\Box q \longrightarrow q) \longrightarrow q)$ and since every other world except v can see a dead end, $(M, w) \vDash \Diamond\Box p$ if $w \neq v$ and $w \neq w_0$. As for v itself, suppose aiming at absurdity that $(M, v) \nvDash \Box(\Box(\Box q \longrightarrow q) \longrightarrow q)$. Then $(M, u) \vDash \Box(\Box q \longrightarrow q) \& \neg q$ since u is the only world v can see (note that u is not therefore in the worldset of q). For each w_i then, $(M, w_i) \vDash \Box q \longrightarrow q$. But trivially, $(M, w_0) \vDash \Box q$, w_0 being a dead end; so $(M, w_0) \vDash q$. In that case, $(M, w_1) \vDash \Box q$ so $(M, w_1) \vDash q$. By the same reasoning, for every i, $(M, w_i) \vDash q$. Thus the complement of the worldset of q is finite and contains u. Therefore the worldset of q is not a member of G, contradicting Corollary 2.5. Hence $\Diamond\Box p \vee \Box(\Box(\Box q \longrightarrow q) \longrightarrow q)$ holds at all $w \in W_M$. ■

We show now that the characteristic axiom of K*, viz. $\Diamond\Box A \vee \Box A$, is not D-valid, for it gets refuted at some world in some model in D.

Lemma 2.7: *There exists a model M in D such that $(M, w) \nvDash \Diamond\Box A \vee \Box A$.*

Proof. Let $M = (F, V)$ be a model based on the recession frame F defined by the valuation function $V(A) = \emptyset$ (the valuation function for every other sentence-letter, if any, is irrelevant for our purpose here of refuting the characteristic axiom-sequent of K*). Then $M \in D$. Since $(M, w_i) \nvDash A$ for all $i \in \mathbb{N}$, we have $(M, u) \nvDash \Box A$, so $(M, v) \nvDash \Diamond\Box A$. And since $V(A) = \emptyset$ we have $(M, u) \nvDash A$, hence $(M, v) \nvDash \Box A$. ■

Theorem 2.8: VB *is incomplete, i.e. it is not characterized by any class of frames.*

Proof. Let's suppose that $\mathcal{K}$ is a class of frames for VB, i.e. with respect to which VB is sound. Then, by Lemma 2.2 all K*-provable sequents are valid with respect to $\mathcal{K}$. In particular $\vdash \Diamond\Box A \vee \Box A$ is valid with respect to $\mathcal{K}$. By Lemma 2.6 and Lemma 2.7 then, there is a $\mathcal{K}$-valid sequent, viz. $\vDash_{\mathcal{K}} \Diamond\Box A \vee \Box A$, which is not VB-provable. Hence VB is incomplete with respect to $\mathcal{K}$. Since $\mathcal{K}$ was arbitrary, this shows that VB is incomplete with respect to any class of frames for which it is sound, and thus VB is incomplete in an absolute sense. ■

3. SEMANTICS FOR THE LANGUAGE OF SECOND-ORDER LOGIC (LSOL)

Basically, the concept of interpretation in second-order logic is similar to the one in first-order logic. Now I am going to present two distinct kinds of interpretation for the language of second-order logic: the *standard* interpretation for 'real' second-order logic, and then the Henkin (or general) interpretation for second-order logic.

3.1. STANDARD SEMANTICS FOR LSOL

For a full, thorough presentation of the topics covered in this section, see [4], [7], [14], [17]. A standard model for a language of second-order logic is basically the same kind of structure as a model for a first-order language, namely a pair $\langle D, I\rangle$ where D is the domain of the model (a set of objects), and I is an interpretation function that gives evaluation clauses for each logical connective (the same evaluation for every model) and assigns appropriate kinds of object constructed from objects that belong to D to each non-logical symbol in the language. To be more specific about this point it has to be added that standard second-order semantics sides with the semantics for the language of first-order logic in virtue of the fact that the domain of both is of the same type, viz. a set of individual objects. So, by setting a domain D the range of both first-order and second-order variables is thereby settled. The function I will take care, as it were, of the assignment of an appropriate object constructed from objects drawn from D to each non-logical symbol. On the other hand, standard second-order semantics differs essentially from Henkin semantics insofar as only in the case of the latter, and not in the case of the former, one should divide the domain of the interpretation in separate ranges: one for the first-order variables (individual variables) and one for second-order variables (sentential variables, n-place function variables, and n-place predicate variables), for any $n \in \mathbb{N}$.

The Tarski-style standard semantics for a second-order language will consist in an extension of the concept of first-order model for the language of first-order logic along the following lines. A standard model of a language of second-order logic, which contains at least one second-order variable, is a structure $\langle D, I\rangle$, where D is a set of objects, and I is an interpretation function. A *variable-assignment* is a function from each first- and second-order variable to elements drawn from D. Thus, a variable-assignment will assign a member of D to each first-order variable, a function from D^n to D to each n-place function variable, and a subset of D^n to each n-place relation variable. Let's observe that in the standard semantics a variable-assignment for an n-place predicate variable X^n in a language of second-order logic is a function from X^n to the set of all n-tuples drawn from D, i.e. the powerset of D^n.

Let now $\mathbf{M} = \langle D, I\rangle$ be a model and s an assignment on $\mathbf{M}$. The denotation of the n-place function variable $f(t_1, .., t_n)$ in $\mathbf{M}$ under assignment s is the value of the function $s(f^n)$ in $\mathbf{M}$ at the sequence of members of D denoted by each term t_i in $\mathbf{M}$ under s. (The denotation function for terms of the language of second-order logic is straightforwardly obtained from its first-order counterpart.)

Satisfaction will be the same kind of relation between models, assignments and formulae as in first-order logic, and we will get the proper inductive definition for a second-order formula's being true in a model $\mathbf{M}$ under an assignment s by adding the following three new clauses for an atomic second-order formula, a second-order universal quantification over function variables, and a second-order universal quantification over predicate variables, respectively. Thus,

I^S If X^n is an n-place predicate variable and $t_1, .., t_n$ is a sequence of n terms, then $\mathbf{M}$, $s \vDash X^n t_1, .., t_n$ iff the sequence of members of D denoted by each t_i under the assignment s is an element of $s(X^n)$.

II^S $\mathbf{M}$, $s \vDash \forall f \Phi$ iff $\mathbf{M}$, $s' \vDash \Phi$ for every assignment s' that is exactly like s at every variable except possibly f.

III^S $\mathbf{M}$, $s \vDash \forall X \Phi$ iff $\mathbf{M}$, $s' \vDash \Phi$ for every assignment s' that is exactly like s at every variable except possibly X.

In virtue of the inter-definability of $\exists$ and $\forall$ the corresponding clauses for $\exists f$ and $\exists X$ can be easily derived from the clauses (**II**^{S}) and (**III**^{S}) above.

3.2. HENKIN-SEMANTICS FOR LSOL

The second semantics for the language of second-order logic is the Henkin semantics. The distinctive feature of it is that n-place predicate variables and n-place function variables can range over strict subsets of D^n and $D^n \times D$, respectively. In other words, the range of every predicate variable and function variable is a fixed subset of relations and functions on the domain, which may very well not include all the relations and all the functions on D^n and $D^n \times D$, respectively.

A Henkin model is a 4-tuple $\mathbf{M}^H = \langle D, D^*, F, I \rangle$ in which D and I are the domain of the model and an interpretation function for the non-logical vocabulary of the language, respectively. The new items in this new kind of model, viz. D^* and F, are a sequence of sets of relations on D^n, and a sequence of sets of functions on $D^n \times D$, respectively. Thus, for any finite $n \in \mathbb{N}$, $D^*(n)$ is a non-empty subset of the powerset of D^n, and $F(n)$ a non-empty subset of functions from D^n to D. The intuitive idea behind this construction of a Henkin-model is that the n-place predicate variables range over $D^*(n)$ and the n-place function variables range over $F(n)$.

A *variable-assignment* on a Henkin model differs significantly from its counterpart on a standard model. Although it is still a function that maps first-order variables into members of D, it varies essentially from what a variable-assignment is on a standard model with respect to predicate and function variables. Thus, a variable-assignment on a Henkin model maps each n-place predicate variable to a member of $D^*(n)$, which as we already remarked may be a proper subset of the powerset of D^n, and each n-place function variable to a member of $F(n)$, which likewise may be a proper subset of the collection of functions from D^n to D. The remaining part of Henkin semantics is basically the same as the standard semantics, except of course for the new meaning that 'variable-assignment' gets in the Henkin semantics. There are then four new clauses:

I^H Let $\mathbf{M}^H = \langle D, D^*, F, I \rangle$ be a Henkin model and s an assignment on $\mathbf{M}^H$. The denotation of $f(t_1, .., t_n)$ in $\mathbf{M}^H$, s, is the value of the function $s(f^n)$ at the sequence of members of D that are the references of each t_i, $1 \leq i \leq n$, on $\mathbf{M}^H$, under s.

II^H If X^n is an n-place predicate variable and $t_1, .., t_n$ is a sequence of n terms, then $\mathbf{M}^H \vDash X^n t_1, \dots, t_n$ if the sequence of members of D that are the references of each t_i, $1 \leq i \leq n$, on $\mathbf{M}^H$, under s, is an element that belongs to $s(X^n)$.

III^H $\mathbf{M}^H$, $s \vDash \forall f \Phi$ iff $\mathbf{M}^H$, $s' \vDash \Phi$ for every assignment s' on $\mathbf{M}^H$ which agrees with s at every variable except possibly at f.

IV^H $\mathbf{M}^H$, $s \vDash \forall X \Phi$ iff $\mathbf{M}^H$, $s' \vDash \Phi$ for every assignment s' on $\mathbf{M}^H$ which agrees with s at every variable except possibly at X.

Remark 3.1: The whole difference between standard semantics and Henkin semantics can be accounted for in terms of the different meanings that are attached to the phrase 'every assignment' in (**II**^{S}) and (**III**^{S}) on the one hand and (**II**^{H}) and (**III**^{H}) on the other hand. In the case of standard semantics an assignment to an n-place predicate variable and to an n-place function variable makes the variables range over the whole powerset of D^n, and over the collection of all functions from D^n to D, respectively. Whereas in the case of the Henkin semantics the collection of assignments may be restricted to those assignments only that assign members of different $D^*(n)$, where $D^*(n) \subseteq D^n$, and $F(n)$, where $F(n) \subseteq D^n \times D$, to the higher-order variables.

4. EXPLAINING INCOMPLETENESS IN MODAL LOGIC

Our semantics for modal logic is essentially a semantics for second-order monadic predicate logic (with a single binary relation constant R)[4]. If we inspect our definition of validity in a frame ($\vDash_F \Phi$) we see that for Φ to be valid in F it must be true in every world in every model based on F. The phrase "every model based on F" is a universal quantifier over assignments of subsets of W to the sentence-letters of the modal language. And since in the canonical translation of LSML into that language of second-order monadic predicate logic a sentence letter of the former becomes a monadic predicate of the latter, the force of "every model based on F" is intuitively – *no matter what subsets of W are assigned to the corresponding monadic predicates*. Hence, the quantification over models in the modal semantics can be captured by a second-order universal quantifier. For example, the statement

$$\vDash_F \Box(P \,\&\, Q)$$

says that for every $w \in W_F$, every world w can see satisfies P and satisfies Q, no matter what properties (subsets of W_F) are assigned to P and to Q. So, in second-order monadic logic, $\vDash_F \Box(P \,\&\, Q)$ can be written

$$F' \vDash \forall P' \forall Q' \forall w \forall u (Rwu \;\longrightarrow\; P'u \,\&\, Q'u)$$

in which we have changed "$\vDash_F$" into "$F \vDash$" to indicate that the pair $\langle W, R \rangle$ is being

[4] Here I follow [10], [12], [13].

regarded as an interpretation for a second-order language with a single binary relation constant R.

To carry out the details of this reductive argument we have to show how the language of sentential modal logic can be mapped into the language of second-order logic. To this purpose we need a collection of recursive rules of translation (schemata) that will take formulae (wffs) of the language of sentential modal logic (LSML) as input and will yield the corresponding formulae of the language of second-order logic (LSOL) as output. What we look for here is a language in which the translation that is carried over is instrumental for the explanation that is sought here, viz. incompleteness in modal logic as a second-order phenomenon. That also backs the concept that modal frame incompleteness is a kind of exemplification of second order incompleteness. And it turns out that what we need is a second-order language that for obvious reasons will be called the language of canonical translation (LCT). In a few words, what we are after here is the bringing about of a mechanism that will allow us to recast the whole apparatus needed to prove the incompleteness of the system VB into the terms that are proper to second-order logic.

The *lexicon of LCT*: One individual variable w; no individual constants; a sentence letter $\wedge$; for each sentence letter π of LSML except $\wedge$, the corresponding monadic-predicate letter λ_π; for each sentence letter π of LSML except $\wedge$, the corresponding monadic-predicate variable τ_π; sentential connectives, second-order and first-order quantifier symbols $\forall^2, \exists^2, \forall, \exists$, and parentheses. The syntax of LCT:

f-at: $\wedge$ is an atomic wff; if λ is any predicate letter and τ any predicate variable then λw, τw are atomic wffs;

f-con: If Φ and Ψ are wffs then so are $\neg\Phi$, $\Phi \,\&\, \Psi$, $\Phi \vee \Psi$, $\Phi \longrightarrow \Psi$, and $\Phi \equiv \Psi$;

f-q^1: If Φ is a wff, then $\exists w\Phi$, and $\forall w\Phi$ are wffs;

f-q^2: If Φ is a wff, then $\exists^2\tau_\pi\Phi$, and $\forall^2\tau_\pi\Phi$ are wffs;

f!: Nothing is a wff unless it is certified as such by the previous rules.

The recursive schemata for translating modal formulae of LSML into LCT:

Trans^2-at: $\mathrm{Trans}^2[\wedge, v] = \wedge$, where v is a fixed first-order variable;
$\mathrm{Trans}^2[\pi, v] = \lambda_\pi v$, if π is a sentence-letter in LSML other than $\wedge$ and λ_π is the prime predicate corresponding to the sentence-letter π;

Trans$^2 - \neg$: $\mathrm{Trans}^2[\neg\Phi, v] = \neg\mathrm{Trans}^2[\Phi, v]$;

Trans$^2 - \&$: $\mathrm{Trans}^2[\Phi \,\&\, \Psi, v] = (\mathrm{Trans}^2[\Phi, v] \,\&\, \mathrm{Trans}^2[\Psi, v])$;

Trans$^2 - \vee$: $\mathrm{Trans}^2[\Phi \vee \Psi, v] = (\mathrm{Trans}^2[\Phi, v] \vee \mathrm{Trans}^2[\Psi, v])$;

Trans$^2 - \longrightarrow$: $\mathrm{Trans}^2[\Phi \longrightarrow \Psi, v] = (\mathrm{Trans}^2[\Phi, v] \longrightarrow \mathrm{Trans}^2[\Psi, v])$;

Trans$^2 - \equiv$: $\mathrm{Trans}^2[\Phi \equiv \Psi, v] = (\mathrm{Trans}^2[\Phi, v] \equiv \mathrm{Trans}^2[\Psi, v])$;

Trans$^2 - \Box$: $\mathrm{Trans}^2[\Box\Phi, v] = \forall v'(Rvv' \longrightarrow (\mathrm{Trans}^2[\Phi, v']))$;

Trans$^2 - \Diamond$: $\mathrm{Trans}^2[\Diamond\Phi, v] = \exists v'(Rvv' \,\&\, (\mathrm{Trans}^2[\Phi, v']))$.

To get the second-order sentence counterpart of a modal sentence we apply the schemata Trans^2 from outside in. Thus, where Φ_μ is any sentence in LSML, we start by an application of the appropriate Trans^2 to the main connective of Φ_μ, and then at every subsequent step we apply appropriate Trans^2 schemata to the main connectives of each formulae thereby obtained. We stop the translation after Trans^2 has been applied to every atomic sentence letter that occurs in Φ_μ. It is worth observing that in $(\mathrm{Trans}^2 - \Box)$ and $(\mathrm{Trans}^2 - \Diamond)$, a new meta-variable v' occurs. Just for getting a unique translation for a necessitate or possibilitate formula one can make the stipulation that there is a specific order in which such variables are to be picked up when those two Trans^2 clauses are applied, e.g. first u, then v, then u', then v', and so on.

The result of these applications of Trans^2 will be an open sentence of LCT, with the predicate variables that correspond to sentence letters in Φ_μ free. Thus, if Φ_μ is $\Diamond(A \vee B)$, and the predicate-variables that correspond to A and B are X, and Y, respectively, then after obvious successive applications of Trans^2 what we get is the open second-order sentence Φ_σ with X and Y free: $\exists u(Rwu \,\&\, (Xu \vee Yu))$.

Now, Φ_σ^* is the *full* second-order translation of Φ_μ, $\mathrm{Fsot}[\Phi_\mu]$ for short, iff Φ_σ^* is the universal closure of Φ_σ with respect to all free first- and second-order variables of $\mathrm{Trans}^2[\Phi_\mu, w]$, and $\Phi_\sigma = \mathrm{Trans}^2[\Phi_\mu, w]$. In symbols,

$$\mathrm{Fsot}[\Phi_\mu] = \Phi_\sigma^* = \forall p_1 .. \forall p_n \forall w \mathrm{Trans}^2[\Phi_\mu, w],$$

where $p_1, .., p_n$ are the monadic predicate variables (second-order variables) corresponding to the sentence letters $\pi_1, .., \pi_n$ which occur in Φ_μ.

Using this recursive procedure we can get the Fsot of the characteristic axiom-sequents of VB and K*, respectively.

$$\mathrm{Fsot}[\Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B)] = \forall X \forall Y \forall w\{\exists u(Rwu \,\&\, \forall v(Ruv \longrightarrow Xv) \vee \forall u(Rwu \longrightarrow ([\forall v(Ruv \longrightarrow (\forall v'(Rvv' \longrightarrow Yv') \longrightarrow Yv))] \longrightarrow Yu))\};$$

$$\mathrm{Fsot}[\Diamond\Box A \vee \Box A] = \forall X \forall w\big[\exists u\big(Rwu \,\&\, \forall v(Ruv \longrightarrow Xv)\big) \vee \forall u(Rwu \longrightarrow Xu)\big]$$

However, it is not only formulae of LSML that have to be mapped into corresponding formulae of LSOL. For to carry out the attempted explanation of modal incompleteness we also need a way of reconfiguring the modal possible world semantics and the main metalogical modal notions definable within that frame as a second-order semantics, and second-order metalogical notions, respectively. It is worth keeping in mind that with respect to modal languages two different modal semantic systems can be constructed, viz. one which is based on the notion of *Kripke-frame*, and a second one which is based on the notion of *General-frame*. The main modal concept of interest for the issue of completeness vs. incompleteness, viz. the notion of a *modal formula's being valid in a frame*, gets the well-known definition "true in every world in every model based on a given frame". And of course, the definition will differ according to whether the frame in question is a *Kripke-frame* or a *General-frame*. Now we show how to reconfigure the modal semantics as second-order semantics.

If F_K is a *Kripke-frame* for the sentences of LSML, define S_2, the second-order model-structure corresponding to F_K, as the model-structure S2 whose domain D is the domain W_F of F_K , which assigns to the n-place predicate letter (constant) Acc a set of n-tuples of objects drawn from D^n that corresponds exactly to the n-tuples of worlds that F_K assigns to R_{F_K}. (Hence Acc has the same degree as R_{F_K}.) Accordingly, for any *Kripke-model* which is based on a *Kripke-frame*, define the second-order model corresponding to the *Kripke-model*, as the interpretation that in addition to the correspondence defined above between *Kripke-frames* and *second-order model-structures* is such that for each sentence-letter π in LSML that is assigned a truth-value by each world in W_K under the evaluation function V, it assigns to the corresponding $\text{Trans}^2[\pi]$ the extension which consists in exactly those $w \in W_K$ such that $w(\pi) = \text{T}$. As readers can very easily check for themselves, the interpretation S_2 thereby obtained is a standard second-order model of the sort defined before in the subsection about the standard semantics for LSOL.

Remark 4.1: The same kind of maneuver allows us to make the transition from a *general-frame* and *model* for LSML to a *Henkin-frame* or *model* for LSOL. The telltale difference between the current case and that worked out one paragraph back is the following. In addition to what we had before, here we need to map the set G of sets of worlds drawn from the domain W of a *general-frame* into the set D_H^* of subsets of the domain D_H of the second-order Henkin-model-structure. Then, as before, we let the valuation-function I_H assign each $\text{Trans}^2[\pi]$ the extension which consists in exactly those $w \in W$ such that $w(\pi) = \text{T}$. Obviously, this amounts to an assignment of a set of n-tuples that belongs to D_H^* to each $\lambda_\pi \in$ LSOL, which mirrors exactly the modal counterpart where sets of worlds drawn from F_G are assigned under V_G to every $\pi \in$ LSML. I am now in the position to state a result, which is needed for the explanation that I seek in this paper.

Theorem 4.2: *For any sentence $\Phi_\mu \in$ LSML there exists a unique corresponding sentence $\Phi_\sigma \in$ LSOL such that $\Phi_\sigma =$ Fsot[Φ_μ], and for any* Kripke-frame $F_K = \langle W, R\rangle$ *there exists a corresponding standard second-order model-structure $S_2 = \langle D, Acc\rangle$ such that $\langle W, R\rangle \vDash_G \Phi_\mu$ iff $\langle D, Acc\rangle \vDash_{S_2} \Phi_\sigma$.*

Further, for any sentence $\Phi_\mu \in$ LSML there exists a corresponding sentence $\Phi_\sigma \in$ LSOL *such that $\Phi_\sigma =$ Fsot[Φ_μ], and for any* general-frame $F_G = \langle W, R, G\rangle$ *there exists a corresponding* second-order Henkin-model-structure $H_2 = \langle D, D^*, \text{Acc}\rangle$ *such that $\langle W, R, G\rangle \vDash_G \Phi_\mu$ if and only if $\langle D, D^*, \text{Acc}\rangle \vDash_H \Phi_\sigma$.*

Proof. By Trans^2 recursive schemata, Fsot, and induction. ∎

Remark 4.3: The idea now is to represent the frame incompleteness of VB as a kind of exemplification of the incompleteness of second-order logic with standard interpretation, which allows Fsot[VB] $\vDash_2$ Fsot[K*], even though, in a sense to be made more precise, Fsot[VB] $\nvdash_2$ Fsot[K*]. The whole modal frame incompleteness phenomenon also indicates a weaker expressive power of LSML as opposed to LSOL.

The apparatus of translation will allow us to reconfigure the modal semantic consequence relationship shown above as holding between VB and K* as the following claim of second-order semantic consequence.

Lemma 4.4: Fsot[VB] $\vDash_2$ Fsot[K*], i.e.

$$\begin{aligned}&\forall X\forall Y\forall w\{\exists u(Rwu \mathbin{\&} \forall v(Ruv \longrightarrow Xv)) \vee \\ &\quad \forall u(Rwu \longrightarrow ([\forall v(Ruv \longrightarrow (\forall v'(Rvv' \longrightarrow Yv') \longrightarrow Yv))] \longrightarrow Yu))\} \\ &\quad \vDash_2 \forall X\forall w[\exists u(Rwu \mathbin{\&} \forall v(Ruv \longrightarrow Xv) \vee \forall u(Rwu \longrightarrow Xu)].\end{aligned}$$

Proof. We show ¬Fsot[K*] $\vDash_2$ ¬Fsot[VB]. Unsurprisingly, the argument recapitulates the proof of the Lemma 2.2. Some familiarity with this proof, some feeling of déja vu is to be expected, for the second-order argument that follows here conveys the same general idea of the proof, which we gave before, of VB's modal incompleteness. Using quantifier shift, modality shift, and truth-functional equivalences, Fsot[¬K*] is equivalent to

$$\exists P\exists w[\forall u(Rwu \longrightarrow \exists v(Ruv \mathbin{\&} \neg Pv)) \mathbin{\&} \exists u(Rwu \mathbin{\&} \neg Pu)], \tag{4.1}$$

while Fsot[¬VB] is equivalent to

$$\begin{aligned}&\exists P\exists Q\exists w\{\forall u(Rwu \longrightarrow \exists v(Ruv \mathbin{\&} \neg Pv)) \mathbin{\&} \exists u(Rwu \mathbin{\&} \\ &\quad ([\forall v(Ruv \longrightarrow (\forall v'(Rvv' \longrightarrow Qv') \longrightarrow Qv))] \mathbin{\&} \neg Qu))\}\end{aligned} \tag{4.2}$$

To obtain (4.2), we need a suitable instance, i.e., suitable P_0, Q_0, and w_0. Let P_0 and w_0 be any property and world which yield a true instance of (4.2). Then straight away we have the first conjunct in the body of (4.2). We now have to find a Q_0 and an u_0 such that

$$Rw_0u_0 \mathbin{\&} ([\forall v(Ru_0v \longrightarrow (\forall v'(Rvv' \longrightarrow Q_0v') \longrightarrow Q_0v))] \mathbin{\&} \neg Q_0u_0) \tag{4.3}$$

From our true instance of (4.1) we have $\exists u(Rw_0u \mathbin{\&} \neg P_0u)$ hence there must be an u_0 such that Rw_0u_0. This gives us the first conjunct of (4.3). Let Q_0 be (a property whose extension is) the set $W - \{u\}$. Then obviously we have the last conjunct of (4.3). To obtain

$$\forall v(Ru_0v \longrightarrow (\forall v'(Rvv' \longrightarrow Q_0v') \longrightarrow Q_0v)) \tag{4.4}$$

assume Ru_0v_0. Then, either (a) $u_0 = v_0$, or (b) $u_0 \neq v_0$. If (a) then $\forall v'(Rv_0v' \longrightarrow Q_0v')$ is false, since Rv_0u_0 but $\neg Q_0u_0$; hence

$$\forall v'(Rv_0v' \longrightarrow Q_0v') \longrightarrow Q_0v_0$$

follows. If (b) then again $\forall v'(Rv_0v' \longrightarrow Q_0v') \longrightarrow Q_0v_0$, since Q_0v_0. Thus (4.4) holds, implying that (4.3) holds, and hence by first and second order ∃I, (4.2) holds. ■

However, this result does not really "explain" why every frame for VB is a frame for K*, since it merely restates our earlier proof in a different language. But it allows us to relate the incompleteness of VB to the non-existence of a sound and complete set of inference rules for second-order logic.

Remark 4.5: The point is not that in second-order logic VB $\nvdash_2$ K*. For deductively, there is no one thing which is second-order logic – instead, there are various significantly different sound deductive systems. And though none of them is complete, there is certainly *some* collection of rules determining a deductive consequence relation $\vdash_2$ such that $\mathsf{VB} \vdash_2 \mathsf{K}^*$. For instance, trivially, we could introduce a second-order system of deduction in which the step from any instance of VB to a corresponding instance of K* is a primitive rule. Or, less trivially, inspection of the second-order counterpart of Lemma 2.2, which is already close to a formal deduction, indicates that we will be able to derive Fsot[K*] from Fsot[VB] in predicative monadic second-order logic.

So what, then, is the explanation of the incompleteness of VB?

To make some progress into this issue it is useful to think of modal systems other than K as theories of modality, and K as the logic. Thus, where before we would have written $A \vdash_T \Diamond A$, treating the T-sequent as part of the logic, i.e. as part of the definition of a new deductive consequence relation $\vdash_T$, we will now write instead that

$$A, \Box\neg A \longrightarrow \neg A \vdash_K \Diamond A.$$

This gives rise to a notion of *formula completeness* that is the counterpart of the notion of system completeness:

- A formula σ of LSML is said to be complete iff all the semantic consequences of it and its substitution instances (relative to auxiliary premises and the class of all frames) are derivable from it in K. In symbols: σ is complete iff whenever $\Gamma, \sigma^* \vDash \gamma$ then $\Gamma, \sigma^* \vdash_K \gamma$, where σ^* is a substitution instance of σ.

The result that VB is an incomplete system becomes in this terminology the result that $\Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B)$ is an incomplete formula, because

$$\Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B) \vDash \Diamond\Box A \vee \Box A$$

but

$$\Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B) \nvdash_K \Diamond\Box A \vee \Box A.$$

The explanation of the incompleteness of VB is then (not that there is no second-order logic in which VB entails K*, but rather) that the deductive consequence relation $\vdash_K$ of modal systems (of theories of modality) is significantly weaker than $\vdash_2^p$ in a precise sense:

Definition 4.6: Let L and L^* be two languages and let T (from 'translation') be a function $T: L \longrightarrow L^*$, from the sentences of L into the sentences of L^*. Then $\vdash^*$ is an L^*-consequence relation which is said to be a *conservative extension* of an L-consequence relation $\vdash$, with respect to T, provided the following holds for any set of L-sentences Σ and any L-sentence σ:

$$T(\Sigma) \vdash^* T(\sigma) \text{ only if } \Sigma \vdash \sigma.$$

It follows that a conservative consequence relation is one in the new language which does not allow sequents to be proven unless they are the translation of provable sequents from the original language. Since we already know that

$$\Diamond\Box A \vee \Box(\Box(\Box B \longrightarrow B) \longrightarrow B) \nvdash_K \Diamond\Box A \vee \Box A,$$

and we have just seen that

$$\mathsf{VB} \vdash_2^p \mathsf{K}^*,$$

this shows that $\vdash_2^p$ is non-conservative over $\vdash_K$.

Its extra strength comes in part from the fact that the second-order variables can substitute for, and be substituted by, any first-order formula with one free variable. But it is perfectly conceivable that there should be interpretations with a first-order definable set of worlds that is not the worldset of any modal sentence; for example, in any transitive model where there are two worlds u and v which see and are seen by the same worlds and which make the same atomic sentences true, no modal sentence can have $W - \{u\}$ or $W - \{v\}$ as its worldset. But "$\_ \neq u$" is still a perfectly acceptable first-order formula which is satisfied by all and only the members of $W - \{u\}$. In general, then, in predicative monadic second-order logic we can reason with statements that cannot even be expressed in LSML, which allows us to prove sequents in the former logic which are not provable modally.

The overall moral, then, is that the system VB is uncharacterizable because of the lack of expressive power of LSML as compared to the expressive power of LSOL. Thus, as the case that I worked out in this paper shows, we can reason in predicative second-order logic with formulae that LSML has no power to express. That is the main rationale for there being the case that the sequent Fsot[VB] $\vdash_2^p$ Fsot[K^*] is provable in second-order logic, whereas its modal version, viz. $\mathsf{VB} \vdash_K \mathsf{K}^*$ is not provable modally.

REFERENCES

[1] J. F. A. K. van Benthem. Two simple incomplete modal logics. *Theoria*, XLIV (1): 25–37, 1978.

[2] J. F. A. K. van Benthem. Syntactic aspects of modal incompleteness theorems. *Theoria*, XLV (1): 63–77, 1979.

[3] J. F. A. K. van Benthem. *Modal Logic and Classical Logic*. Bibliopolis, 1983.

[4] G. Boolos, J. P. Burgess, and R. Jeffrey. *Computability and Logic*. Cambridge University Press, 3rd edition, 1989.

[5] G. Boolos and G. Sambin. An incomplete modal logic. *J. Philosophical Logic*, 14: 351–358, 1985.

[6] M. J. Cresswell. Incompleteness and the Barcan formula. *J. Philosophical Logic*, 24: 379–403, 1996.

[7] R. Epstein. *The Semantic Foundation of Logic. Predicate Logic.* Oxford University Press, 1994.

[8] K. Fine. An incomplete logic containing S4. *Theoria*, XL (1): 23–29, 1974.

[9] M. Fitting. Basic modal logic. In D. Gabbay, C. J. Hogger, and J. A. Robinson (eds.), *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 1: Logical Foundations. Clarendon Press, Oxford, 1993, 365–448.

[10] G. Forbes. *An Introduction to Modal Logic*. Tulane University, 1994.

[11] Robert Goldblatt. Fine's Theorem on First-Order Complete Modal Logics. In Mircea Dumitru (ed.), *Metaphysics, Meaning and Modality. Themes from Kit Fine*, Oxford University Press, 2020, 316–334.

[12] G. Hughes and M. Cresswell. *A Companion to Modal Logic.* Methuen & Co. Ltd, 1984.

[13] G. Hughes and M. Cresswell. *A New Introduction to Modal Logic.* Routledge, London, New-York, 1996.
[14] M. Manzano. *Extensions of First Order Logic*. Number 19 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1996.
[15] H. Sahlqvist, *Completeness and correspondence in first and second order semantics for modal logic*, in S. Kanger (ed.), Proc. of the Third Scandinavian Logic Symposium, Amsterdam, 1975, North Holland, 110–143.
[16] K. Segerberg. An essay in classical modal logic. Technical report, University of Uppsala, 1971. 3 volumes.
[17] S. Shapiro. *Foundations Without Foundationalism. A Case for Second Order Logic*. Clarendon Press, Oxford, 1991.
[18] S. K. Thomason. An incompleteness theorem in modal logic. *Theoria*, XL (1): 30–34, 1974.