INDISPENSABILITY AND THE EMPIRICAL CONTENT OF MATHEMATICAL THEORIES

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Abstract. I highlight how questions of indispensability and questions of the empirical content of mathematical theories, although different, can fruitfully inform one another. After elaborating on what it might mean to say that mathematical theories enjoy empirical content, in terms of the modal status of mathematical truths, the confirmation of applied mathematics, and the explanatoriness of applied mathematics, I argue that modal or confirmational terms are not appropriate for establishing whether mathematical theories enjoy empirical content. Where explanation is concerned, I suggest there is a variety of different kinds of mathematical explanation, and so if empirical content of mathematical theories is conceived of in terms of explanation, then there is a variety of ways in which mathematical theories may come to possess empirical content. Consequently, we have to ask which portions of mathematics are indispensable in the best current explanation of which empirical phenomena. In doing so, which theory of explanation is espoused plays a major role in the assessment of whether mathematics is explanatory or merely representational or descriptive.

Keywords: indispesability arguments, mathematical theories, empirical content, explanatory power

§1. INTRODUCTION

In this paper, I will explore what it might mean to say that mathematical theories lack empirical content. I will address the notion of empirical content in three ways: in a possible-worlds semantics, in terms of confirmation, and in terms of explanation. These are not meant to be exact reconstructions of the notion of content, but only aspects of empirical content that mathematical theories may enjoy or lack. I will conclude by suggesting that neither a possible-worlds view nor a confirmation view of content is apt to ascribe empirical content to mathematics, but that empirical content for mathematics might be found by a more careful appraisal of the explanatory aspects of mathematical applications to empirical science.

I will connect the discussion about the empirical content of mathematical theories with the debate about whether mathematics is indispensable for the success of empirical science. Whether mathematics has any empirical content, and whether empirical science contains ineliminable mathematical content, are two different questions. However, I purport to show that the two questions are closely related, and some views about indispensability have important consequences for the empirical content of mathematics.

Quine's seminal work "Two Dogmas of Empiricism" (1950) already contains elements that allow the formulation of an argument purporting to show that mathematics is indispensable for the success of empirical science. Here is a reconstruction of that argument:

> (1) Our best physical theories make an indispensable use of mathematics. – Indispensability
>
> (2) The confirmation of a theory is wholesale, and since theories require much mathematics, confirmation includes the mathematical parts of the theory. – Quine's Confirmational Holism and
>
> (3) Our theories are committed to those things taken to be values of the variables our theories quantify (existentially) over. – Quine's Criterion of Ontological Commitment.
>
> It follows that once we adopt a physical theory, we are committed to the existence of mathematical objects. (Bangu 2008, fn.1)

As Bangu notes, if all premises are granted, it follows that belief in the truth of the currently best physical theories commits the believer to the existence of abstract mathematical objects quantified over in those theories. Moreover, premise (2) states that "confirmation includes the mathematical parts of the theory", so the mathematical parts of an empirical theory come to enjoy just as much empirical content as the theoretical parts of empirical science. This *symmetry* between scientific realism and Platonism (challenged by Sober 1993, p. 37) can be seen as a consequence of the absence of an analytic-synthetic distinction (Quine 1950), because one can no longer claim that truths of mathematics are analytic, whereas truths of physics are synthetic, since they all enjoy the same confirmation from observation. If this is correct, then, *in Quine's view*, the empirical support of mathematical theories stands or falls with the indispensability of mathematical portions of theories of empirical science.

§2. MODAL VIEWS OF MATHEMATICS AND EMPIRICAL CONTENT

Many have been reluctant to share Quine's premises. For example, commitment to *abstracta* is not favored by nominalists. So identifying *loci* of disagreement would be served if the same indispensability conclusion could be arrived at from weaker premises. One especially attractive move has been to draw a distinction between necessary and contingent that could also distinguish mathematics from natural science. Whereas it is logically possible for the universe to have had different laws or to have evolved differently, Cresswell (2006, p. 143) explicitly states that "all mathematical truths are true in all possible worlds". If this were true, and if empirical content were contingent, then no mathematical theory could ever enjoy empirical content. But this reasoning is too quick.

I would like to raise a worry about the claim that all mathematical truths are necessary, which seems to be doing most of the work. Consider the first-order theory K2 of densely ordered sets with neither first nor last element (Mendelson 2010, ex. 2.67, p. 91). In spite of its name, K2 carries no specific commitment to sets, but only to whatever satisfies its set of axioms in any given first-order model. Here is a feature of K2. Because of its density axiom (h), K2 is modeled by the rational numbers (Mendelson, p. 106). However, if axiom (h) were replaced with its negation, the new axiomatic structure – call it K2' – would be modeled by the integers. So here is a case where we have a proposition (h) true at some worlds and false at others, that makes a difference for what objects satisfy K2 and K2'. So it seems to be false that (in general) all mathematical truths are truths that hold in all possible worlds[1].

What I wish to suggest by the K2 example is not that modal approaches to mathematical truth and existence are untenable, but only that they would require considerable elaboration before they could serve to *sever* the connection between mathematical portions of an empirical theory and the confirmation or disconfirmation of that theory. If so, then (a) the affirmation or denial of the claim that mathematical theories enjoy empirical support has to be sought elsewhere, and (b) the notion of empirical content of mathematical theories could be explained differently.

§3. CONFIRMATION AND THE EMPIRICAL CONTENT OF MATHEMATICAL THEORIES

Perhaps the empirical content of mathematical theories (if any) could be cast in terms of empirical confirmation. That is, a mathematical theory would enjoy empirical content inasmuch as it would be able to be confirmed in empirical testing. This would happen whenever a mathematical theory would be applied as part of a theory belonging to empirical science in such a way that that empirical theory could not receive a mathematics-free formulation that was as virtuous (adequate, parsimonious, simple or explanatory) as the mathematical formulation. Seeing empirical content of mathematical theories this way is, again, a way to connect the issue of mathematical empirical content with the issue of indispensability of mathematics to empirical science.

If confirmational holism is completely renounced, this prevents mathematical theories from being seen as receiving support via empirical confirmation of their physical, biological, etc. applications. But as Vineberg (1996) points out, this in itself does not imply that mathematical theories have no empirical content. Rather, the issue of their content is left undecided.

However, if, in addition to rejecting confirmational holism, a symmetry assumption is endorsed, then the conclusion that mathematical theories lack empirical content follows. Here is how I would summarize an argument present in Sober (1993, pp. 49–51):

[1] Notice (h) is a closed formula and is either true or false for any model of first-order logic. (h) says that: "$(\forall x_1)(\forall x_2)(x_1 < x_2 \Rightarrow (\exists x_3)(x_1 < x_3 \wedge x_3 < x_2))$".

(1) A theoretical hypothesis (mathematical or otherwise) can be confirmed only if it can be disconfirmed as well. That is, we can say an observation supports a hypothesis only if had the complementary observation obtained, the hypothesis would have been disconfirmed.

(2) When observations disagree with theories of mathematical physics (or biology, etc.), we do not blame the mathematics for the empirical theory's disconfirmation.

Therefore,

(3) Mathematical parts of empirical theories are not confirmed by observations confirming the empirical theories they are part of.

What this argument does is enforce, in premise (1), a symmetry between confirmation and disconfirmation, and then say in premise (2) that we do not blame mathematics for disconfirmation, concluding that we should not praise mathematics for confirmation. At a minimum, this shows there is no *necessary* reason why, once one rejects confirmational holism, one should apportion any of the confirmation received by an empirical theory to its mathematically indispensable parts.

If the symmetry condition were imposed, the argument successful, and if empirical content were cast in terms of confirmation and disconfirmation, then mathematical theories would not enjoy empirical content. What I wish to suggest is resisting the assimilation of empirical content to confirmation in testing. Avoiding this assimilation is important because there seem to be other ways in which mathematical theories can enjoy empirical content, namely, via their capacity to explain empirical phenomena that occur and that fall under the purview of theories that include applied mathematics.

This line of thought is also present in Colyvan (2014, p. 72), for whom approaching indispensability arguments through the theory of confirmation misses something essential. Colyvan seems to agree that mathematical truths are not the kind of thing that can be confirmed or disconfirmed, but he points out several uses that mathematics may have when applied:

There are several possibilities here:

(i) Mathematics can demonstrate how something surprising is possible (e.g. stable two-species population cycles).

(ii) Mathematics can show that under a broad range of conditions, something initially surprising must occur (e.g. hexagonal structure in honeycomb).

(iii) Mathematics can demonstrate structural constraints on the system, thus delivering impossibility results (e.g. certain population abundance cycles are impossible).

(iv) Mathematics can demonstrate structural similarities between systems (e.g. missing population periods and the gaps in the rings of Saturn). (Colyvan 2014, p. 72)

So the suggestion put forward is that the explanatory power of mathematics, when applied as part of an empirical theory, may sometimes be a legitimate way to confer empirical content to applied mathematics. If this is the case, then even if the conclusion (3) of the argument above were true, mathematical theories could still enjoy empirical content via their explanatory power.

§4. VARIETIES OF MATHEMATICAL EXPLANATION

If inquiry into the empirical content of mathematical theories is to proceed via the explanatory role that may play in some of its empirical applications in which it is indispensable, some clarification is in order concerning what it might mean to ascribe indispensability to mathematical portions of an empirical account of phenomena. Quine (cited in Maddy 1992, p. 278) argues that, from his standpoint, mathematical ontology is condoned by the simplifications it affords of, and *applications* it has in, empirical science. But certainly not *any* applied mathematics may be in-principle indispensable, even though it may be extremely helpful computationally or heuristically (Azzouni 1997, p. 194; Batterman 2010, p. 10). And indispensability cannot be read literally. For example, Field (1980, pp. 30–41) argued that some portions of analytic geometry could be dispensed with in a formulation of the Newtonian mechanics (and captured back by representation theorems). Yet such a procedure would be less explanatory than the ordinary way of presenting Newtonian mechanics, namely, by using the powerful resources of full Hilbert-style geometry. So, as Baker (2009, p. 613) remarks, some mathematical applications are indispensable in the sense of being *more explanatory* than any mathematics-free alternative explanations.

This immediately raises the question of what it might mean for mathematical applications to assume a genuinely explanatory role[2]. If, by the preceding section, mathematical theories may acquire empirical content by explaining empirical phenomena, then the existence of a variety of explanations entails the existence of a variety of ways mathematical theories may acquire empirical content. Here are three quite different cases.

First, Baker (2005, pp. 231–233) argues for the explanatory role of applying a number-theoretic theorem (life-cycle periods of cicadas that equal a prime number of years minimize their intersection with the life-cycle periods of cicada predators). Suppose, contra Saatsi (2011, pp. 144–145), that Baker were correct in the role mathematics plays in accounting for the length of cicada life-cycle periods. Biologists could be quite content to import a result from mathematics (it would provide a strong prediction but spare the need for testing it unless a blatant counterexample ensues), though, *qua* biologists, they might just as well be oblivious to details of number theory that make the theorem possible, e.g., that first-order Peano arithmetic has no finite models because of the mathematical induction axiom

[2] For example, fruitful generalizations and proofs are often said to be explanatory (Mancosu 2008, p. 144), where fruitfulness not only gives formal manipulability, but also "the idea of a proof" in mathematical reasoning, or, in empirical science, omitting causal details about how the process modeled is supposed to work in a way that is revealing about some basic features of that process (Batterman's (2010, pp. 18–19) asymptotic explanation). Colyvan (2014, p. 71) also approaches generalizability, but through structure reinterpretability, by drawing analogies between structures posited by different branches of empirical sciences as a way to cut short the need for further empirical research. What this variety of approaches suggests is that the topic of mathematical explanation is still underexplored (Mancosu 2008, p. 148), and we may expect to find *different kinds* of mathematical explanation (Batterman 2010, p. 23).

(Mendelson 2010, Proposition 3.6(b), p. 158). In other words, biologists could be satisfied to cherry-pick a number-theoretic result (an application of the concept of primeness to the abstract entities, the numbers 13 and 17) without undertaking ontological or conceptual commitments to any other parts of number theory (Azzouni 1997, p. 207). *If* this is the more parsimonious picture of the cicada example, it would be difficult to say number theory enjoys empirical content, as opposed to just the theorem in question. The resulting picture, with insulated results enjoying empirical content, would do little to support the claim that mathematical theories enjoy empirical content (provided a Quinean semantic holism is rejected).

A second and slightly more complex example borrowed from population biology is given by Colyvan (2014, pp. 64–67). He presents the Lotka-Volterra equations for how the rates of predator and prey populations covary, and notes that, for these equations to be used, differential calculus is indispensable. Here, unlike the cicada case, it is not that one single mathematical result is used, but that a very important fragment of calculus is in play: how to solve differential equations, what the second-order derivative of a function is, that the second-order derivable function has to be continuous, etc. Colyvan's example then supports Batterman's (2010, pp. 2–8) suggestion that *operations* (e.g. functional derivation) can be indispensable, just as concepts and *entities* can be, as intermediary steps between using specific mathematical results, and employing whole mathematical *theories*. That a significant amount of mathematical theory is involved in the Lotka-Volterra equations is undeniable, and seemingly essential to the current formulation of the foundations of population biology[3]. In the Lotka-Volterra example, mathematics enters not only through calculus, but also through the idea of mathematical idealization (Batterman 2010, pp. 16–17). One such important idealization is that the predator and prey population growth rates are held constant in the equations. While such a limit assumption (Batterman 2010, p. 19) can be removed by more advanced models (cf. Colyvan 2014, p. 67), it clearly tells *against* the idea that mathematical indispensability always carries with it a higher degree of theoretical complexity.

A third example comes from mathematical physics. Here is Maddy's formulation:

> For example, the calculus is indispensable in physics; the set-theoretic continuum provides our best account of the calculus; indispensability thus justifies our belief in the set-theoretic continuum, and so, in the set-theoretic methods that generate it; examined and extended in mathematically justifiable ways, this yields Zermelo-Fraenkel set theory. (Maddy 1992, p. 280)

Notice Maddy's inference starts from the indispensability of real calculus to the indispensability of ZF. First, Maddy claims that a mathematical *theory* is indispensable: the calculus. So, if the explanatory power of mathematics accounts for the empirical content acquired by that mathematical theory, this should be a prime

[3] Still, Colyvan (2014, fn.14) makes the telling historical qualification that although Volterra was happy to employ as much mathematics as he thought useful, Lotka resisted use of more robust mathematical resources as much as possible.

example. But Maddy also goes *beyond* that, and claims that set theory is indispensable for mathematical physics. Following Maddy, a naturalist might hold that ZFC is a better option. Alternatively, Feferman (1992, pp. 442–443) claims that virtually all of the mathematics needed for purposes of physical applications can receive counterparts in a predicative development of the foundations of mathematics. Quine (1969) himself was sensitive to how foundations may be simplified, and attempted to carry out various constructions as far as he could: a virtual theory of sets, substitutional quantification, set-theoretic representatives for natural numbers, etc.

But beyond the interesting mathematical question of what the most parsimonious foundations would look like, the conceptual move Maddy makes in the quote above is to say that indispensability *carries over* from a mathematical theory *to the foundations* that make it possible. In other words, Maddy is claiming that indispensability is transitive: if a version of set theory is indispensable for continuum mathematics, and if continuum mathematics is indispensable for mathematical physics, then set theory is indispensable for mathematical physics. If this were true, by parity of reasoning, the empirical content ascribed to mathematical theories would also carry over to their foundations. This would have the startling consequence that the foundations of mathematics would be that mathematical field of inquiry enjoying the most empirical content, as opposed to the more traditional one according to which applied mathematics is the closest to empirical phenomena. While Maddy's point raises an interesting question (a different type of indispensability), the extension to ascribing empirical content to mathematical foundations seems *prima facie* implausible.

To review, I have suggested in this section that there may be a variety of mathematical involvement in empirical theories, involvement that trades on the explanatory role of mathematics, and which would, *via* their explanatoriness, endow portions of mathematics with empirical content. Such different degrees of involvement might vary as widely as the use of one single number-theoretic theorem (in the cicada case), and to the essential use of smaller or large fragments of the calculus in population biology (via the Lotka-Volterra equations) and in mathematical physics. To speak of empirical content for mathematical *theories*, one would have to consider the third, and perhaps also the second, examples.

§5. REPRESENTING AND EXPLAINING

But do such uses of mathematics in the empirical sciences count as genuinely explanatory or as merely representational? A variety of critics have made what seem to be similar points. Assessing the cicada example, Saatsi (2011, p. 152) concludes it shows that a number-theoretic applied relation *represents* a biological fact, rather than explain it. Melia (2000, p. 473) claims that, in physical measurements, numbers *index* quantities rather than number-theoretic relations explaining facts about the quantities measured. Batterman (2010, pp. 11–19) discusses Pincock, and Bueno and Colyvan's views. Pincock argues in favor of the *mapping* account of mathematical application, on which mathematical structures are mapped (partially mapped for Bueno and Colyvan) onto physical structures, and this would be one of the most important ways in which mathematics is applied and could be explanatory.

There are differences among these authors, and objections have been raised to such views (Batterman 2010, pp. 16–19). But perhaps the crucial move that I believe is common to them is a question that can be phrased both in indispensability-talk and in content-talk. If indispensability of mathematical structures amounts to nothing more than their mapping onto physical structures, then an open question is why, and in what circumstances, such mappings may end up being explanatory. If empirical content is acquired by mathematical theories on the basis of their indispensability in currently best explanations of empirical phenomena, then the question of whether mathematical theories enjoy empirical content amounts to looking for mappings between the physical systems and the mathematical structures applied, *and* to *further* claiming that the some of the mappings found are explanatory. Those skeptical about indispensability, or about empirical content of mathematical theories, can then go on to deny the second conjunct. Those friendly to both indispensability and empirical content of mathematics may affirm that same conjunct.

Obviously, how plausible the indispensability-*cum*-explanatoriness claim will be will *vary* with the example chosen, but it will also vary with one's theory of *explanation*. Adepts of an exclusively causal theory of explanation may find it hard to see how mathematical *abstracta* could explain concrete (thereby causal) processes and events. Adepts of a unification theory of explanation may find it difficult to understand why empirical scientists may wish to resist mathematization. Adepts of a pragmatic theory of explanation may wonder why the success of applied mathematics is worth explaining, and hence why indispensability should mean anything more than mathematical application that pays off in empirical terms.

I would like to conclude this paper by tracing back its steps. (1) Throughout the text, I have highlighted how questions of indispensability and questions of the empirical content of mathematical theories, although different, can fruitfully inform one another. (2) I have elaborated on what it might mean to say that mathematical theories enjoy empirical content, in terms of the modal status of mathematical truths, the confirmation of applied mathematics, and the explanatoriness of applied mathematics. (3) I have suggested that modal or confirmational terms are not appropriate for establishing whether mathematical theories enjoy empirical content. (4) I have suggested there is a variety of different kinds of mathematical explanation, and so if empirical content of mathematical theories is conceived of in terms of explanation, then there is a variety of ways in which mathematical theories may come to possess empirical content. (5) Consequently, indispensability arguments can no longer be blanket arguments, so we have to ask which portions of mathematics are indispensable in the best current explanation of which empirical phenomena. (6) In doing so, which theory of explanation is espoused plays a major role in the assessment of whether mathematics is explanatory or merely representational. The area of distinctively mathematical explanation clearly deserves further study (cf. Mancosu 2008, p. 148), including how it might inform theories of explanation in the philosophy of science.

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