PROVABILITY AND SATISFIABILITY. ON THE LOCAL MODELS FOR NATURAL DEDUCTION

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Abstract. This paper discusses the relation between the natural deduction rules of deduction in sequent format and the provability valuation starting from Garson's *Local Expression Theorem*, which is meant to establish that the natural deduction rules of inference enforce exactly the classical meanings of the propositional connectives if these rules are taken to be locally valid, i.e. if they are taken to preserve sequent satisfaction. I argue that the natural deduction rules for disjunction are in no better position than the axiomatic calculi in uniquely determining the intended meaning of disjunction when the local models are used, if a satisfied sequent embeds, as a logical inferentialist should require, a *formal derivability relation*. This happens because, when governed by these rules and without additional semantic assumptions, the disjunction sign still expresses a non-extensional connective, i.e. a connective that, properly understood, has no unique logical characteristic. However, this is not a dead end for the logical inferentialists since both a multiple conclusions formalization of the disjunction operator and a bilateralist one do succeed in restoring the standard meaning of disjunction.

Keywords: natural deduction, local models, extensionality, provability valuation.

1. INTRODUCTION

James W. Garson (2010, 2013) has argued that the natural deduction rules of inference enforce exactly the classical meanings of the propositional connectives if these rules are taken to be locally valid, i.e. if they are taken to preserve sequent satisfaction. I show in this paper why his argument for the *Local Expression Theorem* may be seen as inferentially problematic. I will start by describing the general framework of the relation between logical calculi and their models (section II) and then I shall analyze Garson's argument for the idea that the standard meanings of the classical propositional operators could be read off from the natural deduction rules of inference by using the local models (section III). I will then show, in section IV, that the valuation $\boldsymbol{v}^{\vdash}$ (and by extension $\boldsymbol{v}^{\vee}$) belongs to the local models for the natural deduction rules of the calculus Sv, if a satisfied sequent embeds, as a logical inferentialist should require, *a formal derivability relation*.[1] In section V I explain

[1] I found out after writing this paper that this idea was in fact emphasized by Pelletier and Hazen (2012: 399-400) and the discussion bellow may also be seen as an application of their distinction between *de facto truth preserving subproofs* and *subproofs that embody formally valid reasoning* to the relation between local models and the natural deduction rules in sequent format. The use of the local models makes the derivability relation behave as being *de facto truth preserving* and in this case the natural deduction rules *come tantalizingly close* (p. 399) in determining the meanings of the logical terms.

why Garson's argument for the Local Expression Theorem may be taken as inferentially problematic by discussing the concept of non-extensionality and Carnap's view on non-normal valuations. In section VI, I analyze a possible objection by which one may want to defend the local models and I also clarify the way in which the natural deduction rules in sequent format relate to the meta-theory of the relation of formal derivability in propositional calculi. I end (in section VII) by discussing the way in which a multiple conclusions and a bilateralist formalizations manage to block $\boldsymbol{v}^{\vdash}$ (and $\boldsymbol{v}^{\vee}$).

2. LOGICAL CALCULI AND THEIR MODELS

The idea that the standard propositional calculi for classical logic do not uniquely enforce the standard interpretations of their operators is well known.[2] These calculi are called non-categorical, in a particular sense of categoricity, i.e. they allow for valuations that preserve their soundness, but provide their logical symbols with non-standard meanings. Among the proposed solutions for solving this problem, some authors enforced the logical calculi by adding syntactical instruments of a new kind (e.g. Carnap 1943, Smiley 1996, Rumfitt 1997, 2000), some others imposed semantic constraints on the calculi (Church 1944, Koslow 2010, Bonnay and Westerståhl 2016) while others treated in parallel the relation between logical calculi and the semantics that they express (Garson 2013, Murzi and Topey 2021).

James Garson (2013) pointed out that the logical inferentialist idea that the meanings of the logical constants are determined by a logical calculus is both dependent on the format of the calculus (i.e. axiomatic, natural deduction or sequent calculus) and on the way in which the expressive power of the calculus is encoded (i.e. by deductive, local or global models). Let us briefly introduce these notions in a unitary manner.

A sequent $\Gamma \vdash \Delta$ is composed of sequences Γ and Δ of wffs of the language over which they are defined. If Δ has a single member φ, then $\Gamma \vdash \varphi$ is called an argument. If Γ is empty then $\vdash \varphi$ is called an assertion. An axiomatic system is composed of assertions (each axiom and theorem) and a rule of inference whose premises and conclusion are assertions (usually modus ponens). A natural deduction system is composed of rules of inference whose premises and conclusions are arguments. A sequent calculus is composed of rules of inference whose premises and conclusions are sequents.

The expressive power of these logical calculi may be differently encoded depending on what is taken to be a model of a logical calculus. A model V of a

[2] In particular due to the work of Bernstein (1932), Huntington (1933, 1934), Carnap (1943), Church (1944, 1953), Shoesmith and Smiley (1978), McCawley (1981), Belnap and Massey (1990), Garson (1990, 2010, 2013), Smiley (1996), Rumfitt (1997, 2000), Raatikainen (2008), Murzi and Hjortland (2009), Koslow (2010), Hjortland (2014), McGee (2000, 2015), Bonnay and Westerståhl (2016), Murzi and Topey (2021), Brîncuș (2021, 2024), et al.

logical calculus is a set of valuations $\boldsymbol{v}$, where $\boldsymbol{v}$ is a function that images each wff of the language to one of the members of the set {t, f}. If all $\boldsymbol{v} \in \mathrm{V}$ assign t to all the assertions, provable arguments or provable sequents of the system, then V is a deductive model of the logical calculus. If all $\boldsymbol{v} \in \mathrm{V}$ satisfy the axioms and rules of the logical calculus, then V is a local model (these models will be discussed below). If all the rules of the calculus preserve the V-validity of their premise arguments when passing to the conclusion, then V is a global model.

Garson's (2013) analyses show that the multiple conclusions formalizations of propositional logic uniquely determine the standard semantic meanings of the propositional operators for all the three ways of encoding the expressive power of a logical calculus (deductive, local, global models) while the axiomatic formalizations fail to uniquely determine these meanings no matter which of the three kinds of models are used. The classical natural deduction systems illustrate an interesting situation: if their expressive power is encoded by the global models, then they determine intuitionistic meanings for their operators; if their expressive power is encoded by the local models, then they seem to uniquely determine the standard meanings of the propositional operators as they are defined by the standard classical truth tables. The latter idea is called by Garson (2010:165, 2013:37) *Rule Model Theorem* or, respectively, *Local Expression Theorem*. I discuss in the next two sections Garson's reasons for asserting it and why it may be seen as problematic from an inferential point of view.

3. LOCAL MODELS FOR NATURAL DEDUCTION

In the logical framework defined by the structural rules (hypothesis, monotonicity and cut), the disjunction operator is introduced by the following rules of inference:[3]

v Introduction

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}$$

v Elimination

$$\frac{\Gamma \vdash \varphi \vee \psi \quad \Gamma, [\varphi] \vdash \sigma \quad \Gamma, [\psi] \vdash \sigma}{\Gamma \vdash \sigma}$$

[3] In this paper the Greek letters Γ, Δ are used as meta-variables over sets of wffs of the object language and φ, ψ, σ as meta-variables over wffs of the object language. Garson (2013:8-9) uses the Latin letters G, H as meta-variables over sets of wffs of the object language and A, B, C, D as meta-variables over wffs of the object language. I shall use in this paper the Latin letters A, B, C … as propositional constants of the object language. This aspect is important since for Garson a rule of inference is a set of rule instances obtained by substitution, although he assigns truth values directly to the meta-variables from the meta-language, without using explicitly propositional constants.

The logical calculus Sv comprises the structural rules plus these vI and vE rules. The question in our concern is whether the expressive power of Sv is such that it uniquely determines the intended semantic meaning of 'v' as it is defined by its normal truth table or, in other words, whether Sv is a categorical logical calculus if its expressive power is read by using the local models.

A model is a set of valuations $\boldsymbol{v}$ that assign to each wff of the propositional language one of the values from the set {t, f}. A local model V of a rule is a set of valuations $\boldsymbol{v}$ such that each valuation satisfies the rule, i.e. if it satisfies the inputs of the rule, then it also satisfies the output. An argument of the form '$\Gamma \vdash \varphi$' is satisfied by a valuation $\boldsymbol{v}$ iff $\boldsymbol{v}$ either assigns f to at least one member of Γ or it assigns t to φ. A system of rules Sc –where Sc is the natural deduction system obtained by adding to the structural rules the rules for connectives on any list c drawn from the connectives: &, ⟶, ~, v, ↔– locally expresses property P iff for every model V, V is a local model of the rules of S exactly when V has property P.[4] With these notions introduced, we can new formulate Garson's (2013: 37) claim:

Local Expression Theorem. Sc locally expresses the classical truth tables for the connectives on list c.

What this theorem substantially states is that whenever V is a local model of the rules of Sc and $\boldsymbol{v}$ is a member of V, then $\boldsymbol{v}$ obeys the classical truth tables. In other words, all the valuations $\boldsymbol{v}$ from V are standard valuations if V is a local model of a natural deduction system that is under investigation.

To better understand the content of the Local Expression theorem, let us first clarify when a certain semantic property of an expression is uniquely determined by a rule or by a calculus. Garson's (2013:34) answer is formulated in the following definition:

(Local Expression) A system S *locally expresses* property P iff for every model V, V is a local model of the rules of S exactly when V has property P.

According to this definition, that the property P of models is locally expressed by S means that there is no model V such that V is a local model, but V does not have the property P. That V is a local model of S means that for every valuation v in V, if v satisfies the inputs of the rules of S, then v also satisfies the output of these rules. In other words, that the property P of models is locally expressed by S means that there is no model V such that all valuations in V satisfy the rules of S, but V does not have the property P. Now, if S is Sv and P is the semantic property of the disjunction sign of being false when its both disjuncts are false (i.e. Dj4-property), then that the Dj4-property is locally expresses by Sv means that there is no model V such that all valuations in V satisfy the rules of Sv, but V does not have the Dj4- property. That V does not have the Dj4-property means that there is at least one valuation in it that satisfies the rules of Sv, but does not have the Dj4-property. Thus, I emphasize that identifying one single valuation $\boldsymbol{v}$ from the local model V which is non-standard will suffice to show the limits of the local models. My attention below will focus on the system Sv and on its local models.

[4] In Carnap's (1943:3) terminology, the property P is in this case fully formalized by the natural deduction propositional calculus.

Garson's (2013: 38) argument for the idea that Sv is categorical aims to show that the last row of the normal truth table for disjunction is enforced by the natural deduction rules if the local models are used. The argument runs as follows:[5]

> Assume that $\boldsymbol{v}$(A)=$\boldsymbol{v}$(B)=f, and show $\boldsymbol{v}$(AvB)=f as follows. By definition, valuation $\boldsymbol{v}$ assigns f to at least one wff D. Since $\boldsymbol{v}$ satisfies (v Elimination), we have that if $\boldsymbol{v}$ satisfies the arguments / AvB, A / D, and B / D, then $\boldsymbol{v}$ satisfies / D. But $\boldsymbol{v}$(D)=f, so $\boldsymbol{v}$ cannot satisfy / D, with the result that $\boldsymbol{v}$ must fail to satisfy one of the arguments / AvB, A / D, and B / D. However, $\boldsymbol{v}$(A)=$\boldsymbol{v}$(B)=f, and so $\boldsymbol{v}$ satisfies both A / D and B / D. So $\boldsymbol{v}$ fails to satisfy / A v B, and $\boldsymbol{v}$(AvB)=f as desired.

What Garson assumes in this argument is that every valuation $\boldsymbol{v}$ is a consistent valuation and by this he means that $\boldsymbol{v}$ assigns the value f to at least one wff. In other words, the trivial valuation that assigns t to all wffs of the propositional language is excluded by semantic stipulation.[6] What Garson further assumes, without any explicit justification, is that the conclusion of the vE rule is one of those sentences whose value is assigned by the valuation $\boldsymbol{v}$ to be f. *Prima facie*, since at least one sentence of the propositional calculus will be interpreted as being false, then there is no *a priori* reason for not taking one of these sentences to be the target formula in the vE rule. With these assumptions at work, Garson shows that when a valuation $\boldsymbol{v}$ assigns the value f to both the disjuncts of a disjunction and to the conclusion derivable from them, then the disjunction itself will be false on $\boldsymbol{v}$.

Garson's (2010:167; 2013:39) more general explanation for the success of the natural deduction rules when local models are used is granted to the assignment of the value f to a subformula of a molecular sentence:

> They manage to fix the formerly missing row of the truth table for a connective c because a subformula (A and/or B of the wff AcB appears as a hypothesis in some input. When $\boldsymbol{v}$ is a local model of such a rule, a condition of the form: 'if $\boldsymbol{v}$(A)=f then …' is enforced on $\boldsymbol{v}$. […] Defining rules over arguments with subformulas in their hypotheses is the secret to success.

This explanation works in the case of vE rule as used in Sv, but only is we assume that the target formula derived from each disjuncts is mapped by the valuation $\boldsymbol{v}$ into f. If we consider an axiomatic system for classical propositional logic, the provability valuation $\boldsymbol{v}^{\vdash}$ that assigns t to a wff if and only if it is logically provable satisfies both the axioms and the rule of inference, since the axioms are logically provable from the empty set and the premises of modus ponens are not

[5] The symbol '/ ' belongs to the object language, while the symbol '⊢' belongs to the metalanguage. 'Γ⊢φ' expresses the claim that the argument 'Γ / φ' from a given system is derivable in that system. For this convention see Hacking (1979:292) and Garson (2013:9). The letters A, B, D are used by Garson as metavariables for wffs of the object language (see footnote 2).

[6] This semantic assumption has been adopted by several authors, including: Beth (1963: 490), McGee (2000:71), Bonnay and Westerståhl (2016: 725).

logically provable and thus mapped by $\boldsymbol{v}^{\vdash}$ into f. However, $\boldsymbol{v}^{\vdash}(A)=\boldsymbol{v}^{\vdash}(\sim A)=f$, while $\boldsymbol{v}^{\vdash}(A \vee \sim A)=t$.[7] Thus, the axiomatic propositional calculi allow for non-standard local models V, since $\boldsymbol{v}^{\vdash} \in V$. The question that remains to be answered is whether Sv allows for non-standard local models.

4. THE PROVABILITY VALUATION IS PART OF THE LOCAL MODELS FOR THE CALCULUS SV

Garson's reasoning in the case of Sv seems to go too fast to the desired conclusion. Let us note that Sv is categorical exactly when *all* the valuations that are compatible with the rules, i.e. that satisfy the rules, determine the standard semantic meaning of disjunction. Moreover, we should pay attention to the fact that there are instances of vE rule, although not in Sv, in which the formula derived from both disjuncts is a logical theorem, as in this particular case:

$$\frac{\begin{array}{l}\vdash A \vee \sim A \\ [A] \vdash (A \longrightarrow B) \vee (B \longrightarrow A) \\ [\sim A] \vdash (A \longrightarrow B) \vee (B \longrightarrow A)\end{array}}{\Gamma \vdash (A \longrightarrow B) \vee (B \longrightarrow A)}$$

When this instance of the vE rule is considered, the provability valuation ($\boldsymbol{v}^{\vdash}$) that assigns t only to theorems and f to non-theorems will satisfy the rule, since every rule input is satisfied and the rule output is also satisfied –since their conclusions are logical theorems. Thus, $\boldsymbol{v}^{\vdash}$ is a local model of this instance of the vE rule. However, in this valuation both A and ~A are false while Av~A is true.

The problem that has to be clarified is whether $\boldsymbol{v}^{\vdash}$ is a member of V, when V is a local model of Sv. We know that V is a local model of a rule R iff every member of V satisfies R. Does $\boldsymbol{v}^{\vdash}$ satisfy Sv? Certainly, $\boldsymbol{v}^{\vdash}$ satisfies the structural rules. It remains to be analyzed whether $\boldsymbol{v}^{\vdash}$ satisfies the vI and vE rules. The vI rules raise no problem since they fix the first three rows of the normal truth table (NTT) for disjunction. The problematic rule is the vE one. Thus, once again: does $\boldsymbol{v}^{\vdash}$ satisfy the vE rule?

Lemma: If $\boldsymbol{v}^{\vdash}$ satisfies the vE rule inputs, then it satisfies its output.

Proof: Consider the vE rule, with $\Gamma=\emptyset$. Suppose $\boldsymbol{v}^{\vdash}$ would satisfy the inputs of the rule, i.e. $\boldsymbol{v}^{\vdash}(\vdash \varphi \vee \psi)=t$, $\boldsymbol{v}^{\vdash}([\varphi] \vdash \sigma)=t$, $\boldsymbol{v}^{\vdash}([\psi] \vdash \sigma)=t$. If $\boldsymbol{v}^{\vdash}(\vdash \varphi \vee \psi)=t$, it would follow that $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)=t$. This means that $\varphi \vee \psi$ is a theorem. If $\boldsymbol{v}^{\vdash}([\varphi] \vdash \sigma)=t$, it would follow that $\boldsymbol{v}^{\vdash}(\vdash \varphi \longrightarrow \sigma)=t$. So $\varphi \longrightarrow \sigma$ is a theorem. Likewise, if $\boldsymbol{v}^{\vdash}([\psi] \vdash \sigma)=t$, it would follow that $\boldsymbol{v}^{\vdash}(\vdash \psi \longrightarrow \sigma)=t$. So $\psi \longrightarrow \sigma$ is a theorem. Thus, *due to the deducibility assertions*

[7] This provability evaluation is discussed in more details, in relation to McGee's open-endedness approach for attaining categoricity, in Brîncuș (2021).

that the premise sequents of the vE rule make, if σ deductively follows both from φ and from ψ, then it follows from φvψ. Since φvψ is a theorem, then σ will also be a theorem and, thus $\boldsymbol{v}^{\vdash}(\vdash\sigma)$=t. Therefore, $\boldsymbol{v}^{\vdash}(\vdash\sigma)$ is true and, thus, satisfied.[8]

Theorem: If $\boldsymbol{v}^{\vdash}$ satisfies the v-Elimination rule, then $\boldsymbol{v}^{\vdash}$ is a member of the local model V.

Proof: Suppose that $\boldsymbol{v}^{\vdash}$ is not a member of V. This means that $\boldsymbol{v}^{\vdash}$ does not satisfy the v-Elimination rule, i.e. $\boldsymbol{v}^{\vdash}$ satisfies the inputs of the rule, but it does not satisfy its output. However, by the lemma above, $\boldsymbol{v}^{\vdash}$ satisfies the vE rule. Therefore, $\boldsymbol{v}^{\vdash}$ is a member of V.

Observation*:* The proof of the Lemma uses the additional proof-theoretic premise that a formula which deductively follows from each one of two sentences, then it deductively follows from their disjunction. In other words, the deducibility background assures us that if there is a derivation D_1 leading from φ to σ and there is also a derivation D_2 leading from ψ to σ, then there is also a derivation D_3 leading from φvψ to σ. More generally, it assumes that the deducibility relations stated in the premise and conclusion are formal logical relations. The use of this additional premise will be discussed and justified below, in section VI.

As we shall understand better in the next section, Garson's argument for Sv is correct, but it is insufficient for showing that the Sv calculus uniquely determines the normal meaning of disjunction when the local models are used. What he shows is simply that *there are some situations in which the rules, indeed, enforce the standard meaning of disjunction (namely, when the class of valuations is restricted to those that assign falsity to the target formula of the vE rule)*. We shall see in a moment that Carnap (1943) was perfectly aware of this phenomenon of non-extensionality. The v-Elimination rule is compatible with a valuation $\boldsymbol{v}^{\vdash}$ that assigns f both to φ and to ψ, and assigns t to φvψ and whatever follows from φvψ, since when it assigns t to φvψ, i.e. φvψ is a theorem, it also assigns t to σ, i.e. σ is also a theorem, in virtue of the deducibility relations (this aspect will be clarified in section VI below).

5. CARNAP ON NON-NORMALITY AND NON-EXTENSIONALITY

Carnap (1943) analyzed the question whether the standard calculi of propositional and predicate logics, i.e. calculi with a finite number of premises and a single conclusion, uniquely determine the standard meanings (normal meanings in Carnap's terms) of their logical symbols. Carnap considered the standard axiomatic calculi of classical propositional logic and he took as benchmark for the expressive power of the calculi what Garson calls 'deductive models'. For the propositional calculi,

[8] I use the turnstyle (⊢) instead of the slash (/) since the vE rule is formulated in sequent format and, thus, it is taken to be a metarule.

Carnap proved that there are two kinds of non-normal interpretations, one in which all sentences are true and one in which at least one sentence is false. What Carnap (1943:75) interestingly pointed out is that when a syntactical logical symbol is non-normally interpreted by a valuation, it is not the case that the symbol has another logical characteristic (i.e. truth table), but rather that it has no truth tables, i.e., it is non-extensional:

> Let *K* contain PC_1 or PC_1^D and a_k be a sign for the connection ${}_cConn_r^2$ in K. Let *S* be a true interpretation of *K* such that the following is the case (provided that this is possible; that will be discussed later): a_k is not a sign for $Conn_r^2$ in *S* and hence has a *non-normal interpretation* in S (DI). Then at least one rule for $Conn_r^2$, represented by a line in the truth-table for this connection, will be violated by a_k in *S* in at least one instance, i.e. with respect to at least one pair of closed sentences as components. This violation of a normal truth-table by a_k is not necessarily such that a_k has another truth-table in *S*. Let us suppose that a certain rule for $Conn_r^2$ in NTT states the value F for the distribution TF of the components. Then it may happen that for some instance with the values TF the full sentence of a_k is indeed false, while for another instance with the same values TF it is true. If this happens, a_k has no truth-table in *S*, neither the normal nor another one; the truth-value of a full sentence of a_k is not a function of the truth-values of the components; a_k is *non-extensional* (DI2-2, T12-6).

PC_1 and PC_1^D are two axiomatic systems for classical propositional logic that Carnap considers in his book and 'a_k' is a logical symbol of the calculus K. If a_k is taken to be the syntactical sign for disjunction, then there is at least one pair of closed sentences φ and ψ such that both $\boldsymbol{v}^{\vdash}(\varphi)= \boldsymbol{v}^{\vdash}(\psi)$=f, but $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)$=t. Thus the valuation $\boldsymbol{v}^{\vdash}$ disobeys the fourth row of the normal truth table for disjunction with respect to the sentences φ and ψ. However, with respect to the sentences ∼φ and ψ, the valuation $\boldsymbol{v}^{\vdash}$ will obey the Dj4 row.

Proposition: If $\boldsymbol{v}^{\vdash}(\varphi)= \boldsymbol{v}^{\vdash}(\psi)$=f and $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)$=t, then $\boldsymbol{v}^{\vdash}(\sim\varphi)= \boldsymbol{v}^{\vdash}(\psi)= \boldsymbol{v}^{\vdash}(\sim\varphi \vee \psi)$=f.

Proof: Consider the following sequent $\{\varphi \vee \psi;\ \sim\varphi\} \vdash \psi$. Since $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)$=t and $\boldsymbol{v}^{\vdash}(\psi)$=f, then it follows that $\boldsymbol{v}^{\vdash}(\sim\varphi)$=f. Likewise, $\{\varphi \vee \psi;\ \sim\psi\} \vdash \varphi$; but since $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)$=t and $\boldsymbol{v}^{\vdash}(\varphi)$=f it follows that $\boldsymbol{v}^{\vdash}(\sim\psi)$=f. Therefore, if φ and ψ are both false and φvψ is true, then $\boldsymbol{v}^{\vdash}(\sim\varphi)= \boldsymbol{v}^{\vdash}(\sim\psi)$=f. However, $\{\varphi \vee \psi;\ \sim\varphi \vee \psi\} \vdash \psi$; since $\boldsymbol{v}^{\vdash}(\varphi \vee \psi)$=t and $\boldsymbol{v}^{\vdash}(\psi)$=f, it follows that $\boldsymbol{v}^{\vdash}(\sim\varphi \vee \psi)$=f.

Therefore v obeys Dj4 with respect to ∼φ and ψ, which are both false, although it disobeys Dj4 with respect to φ and ψ. Hence, the disjunction sign is non-extensional. However, this does not mean that it has two truth tables, but simply that it has no logical characteristic (see also Carnap (1943: 79)). Moreover, it cannot be said that there are two disjunction signs v_1 and v_2 since the rules for the disjunction sign allow precisely one inferential role for it, and Carnap (1943:29, Theorem T7-4a) was well aware of this fact.[9]

[9] This idea is better known today as Belnap's uniqueness condition.

The form of an extensional truth condition is formulated by Garson (2013:44) by a function f that expresses some features defined only over the values $\boldsymbol{v}$(A) and $\boldsymbol{v}$(B):

$\boldsymbol{v}$(AcB)=t iff f($\boldsymbol{v}$(A), $\boldsymbol{v}$(B)).

It is quite clear, however, that since a disjunction is sometimes true when its both disjuncts are false, and some other time false when its both disjuncts are false, the syntactical sign v is a non-extensional one.[10] Garson's argument for the *Local Expression Theorem* in the case of the system Sv only shows that the rules for disjunction sometimes make the disjunction sign to behave standardly, namely, when the target formula of the vE rule is false.

The reader may raise doubts concerning the example gave in the *Proposition* above because it involves sentences that are not part of the language of Sv, since Sv has no negation sign. These doubts are justified, but the situation can be easily clarified by means of an example of a valuation that makes the disjunction sign non-extensional in the language of Sv. Consider again the vE rule with Γ=∅:

$$\frac{\vdash A\mathrm{v}B \qquad [A]\vdash C \qquad [B]\vdash C}{\vdash C}$$

and suppose that Dj4 is violated by a valuation $\boldsymbol{v}^{\mathrm{v}}$ with respect to the propositional constants A and B. Let us first suppose that this is possible and then show that $\boldsymbol{v}^{\mathrm{v}}$ is non-empty for Sv. The possibility of $\boldsymbol{v}^{\mathrm{v}}$ entails the following:

i) $\boldsymbol{v}^{\mathrm{v}}$(A)= $\boldsymbol{v}^{\mathrm{v}}$(B)=f and $\boldsymbol{v}^{\mathrm{v}}$(AvB)=t (by supposition).
ii) A is different from B (and conversely). Proof: if A were B, then A, which is derivable from AvA, would be derivable from AvB and, thus, $\boldsymbol{v}^{\mathrm{v}}$(A) would be t. Since $\boldsymbol{v}^{\mathrm{v}}$(A)=f, it follows that A≠B.
iii) For any sentence C that is derivable both from A and from B, $\boldsymbol{v}^{\mathrm{v}}$(C)=t. Proof: If a sentence is derivable both from A and from B, then it is derivable from AvB. Since $\boldsymbol{v}^{\mathrm{v}}$(AvB)=t, then $\boldsymbol{v}^{\mathrm{v}}$(C)=t.
iv) A is not derivable from B, nor B from A. Proof: If A were derivable from B, since A is derivable from A, it would be derivable from AvB (by vE) and, thus, $\boldsymbol{v}^{\mathrm{v}}$(A) would be t. But $\boldsymbol{v}^{\mathrm{v}}$(A)=f. (Likewise for B is not derivable from A.)

[10] For a discussion of the (non)extensionality of the propositional connectives in relation to the distinction between normal and non-normal valuations from a structuralist perspective see Koslow (2010: 130-134).

Since there are no theorems in Sv, i.e. written only in terms of disjunction, the valuation $\boldsymbol{v}^{v}$ will be the analogue of the valuation $\boldsymbol{v}^{\vdash}$ discussed above. $\boldsymbol{v}^{v}$ will assign t or f to the wffs of Sv in the following way:

1. $\boldsymbol{v}^{v}$ (A)= $\boldsymbol{v}^{v}$(B)=f
2. $\boldsymbol{v}^{v}$ (AvB)=t
3. For every C, if AvB⊢C, the $\boldsymbol{v}^{v}$(C)=t.
4. For all the other sentences D, if D≠C, then $\boldsymbol{v}^{v}$ (D)=f. [11]

This example illustrates likewise the idea that the disjunction sign is non-extensional, since $\boldsymbol{v}^{v}$(AvB)=t while $\boldsymbol{v}^{v}$(A)= $\boldsymbol{v}^{v}$(B)=f and $\boldsymbol{v}^{v}$(AvD)=f while $\boldsymbol{v}^{v}$(A)= $\boldsymbol{v}^{v}$(D)=f. Strictly speaking, $\boldsymbol{v}^{v}$ instead of $\boldsymbol{v}^{\vdash}$is the a non-standard local valuation for Sv since the effective applicability of $\boldsymbol{v}^{\vdash}$ to Sv requires that the major premise of the vE rule has the logical form of a theorem and there is no such theorem in the calculus Sv. In Carnap's (1943: 77) terms, although $\boldsymbol{v}^{\vdash}$ satisfies Sv, it has 'no instances of application'. The discussion of the valuation $\boldsymbol{v}^{\vdash}$ for Sv is nevertheless relevant since Sv is only a subsystem of classical propositional logic and $\boldsymbol{v}^{\vdash}$ satisfies the vE rule and has instances of application when the system is extended, for instance, by the system S~.

6. AN OBJECTION AND ITS REPLY

a) Objection.[12] When evaluating the expressive power of the local models for a certain rule, one needs to consider all possible instances of the rule since Garson (2010:161-163) takes a rule to be a set of rule instances and, thus, a valuation satisfies a rule when it satisfies all of its instances. One of the instances that has to be considered is the one Garson (2013:38) is interested in, namely the instance when a formula that is assigned false will appear in the consequent the output of the v-Elimination rule. For example, let us consider the following instance of the v-Elimination rule:

$$\frac{\begin{array}{l}\vdash A\text{v}{\sim}A\\ [A]\vdash C\\ [{\sim}A]\vdash C\end{array}}{\vdash C}$$

[11] For an analysis of the non-categoricity of positive propositional calculus see Brîncuș and Toader (2019:63-64).

[12] Many thanks to a reviewer for raising this objection, which cannot be technically dismissed if all the assumptions used in Garson's reasoning are accepted. However, some of them are inferentially problematic, as the discussion below will show, and, thus, I think that the story is still worth being told.

In this particular case –the objection goes– the valuation $\boldsymbol{v}^{\vdash}$ satisfies the three premise sequents: The first is satisfied because the consequent is a theorem and, thus, $\boldsymbol{v}^{\vdash}$(Av~A)=t. The second and third are satisfied, because neither antecedent is a theorem and, thus, $\boldsymbol{v}^{\vdash}$(A)= $\boldsymbol{v}^{\vdash}$(~A)=f. The conclusion of the rule is not satisfied, since C was supposed to be false. Therefore, the valuation $\boldsymbol{v}^{\vdash}$ does not satisfy the v-Elimination rule and, thus, it is not a member of the local model V.

b) Reply to Objection. This objection closely resembles Garson's (2013:38) reasoning for showing that the v-Elimination rule uniquely determines the fourth row of the classical truth table for disjunction (Dj_4), only that Garson does not consider the major premise to have the logical form of a theorem, since he considers Sv. Garson concludes the unsatisfiability of the major premise from the unsatisfiability of the conclusion.[13] Since C is taken to be false in this instance of the rule, then at least one of the premises has to be unsatisfied for obeying the soundness of the rule, i.e. the preservation of satisfaction. However, $\boldsymbol{v}^{\vdash}$ satisfies all the three premises, but does not satisfy the conclusion. Therefore, the objection concludes that $\boldsymbol{v}^{\vdash}$ does not satisfy all the instances of the v-Elimination rule. I shall make two observations with regards to this objection and to the relation between $\boldsymbol{v}^{\vdash}$ and local validity (A, B).

A. The main problem with this objection, which may seem *prima facie* very plausible, is that it incorporates semantical assumptions into the definition of an instance of a rule, which should be a purely syntactical instrument. The instance of the v-Elimination rule formulated in the objection above is syntactically correct, but it is not part of the definition of an instance of a rule that certain sentences (in this case, the target formula C) should receive a certain truth value. The valuational space associated with Sv could be *a priori* thought of as comprising two exclusive classes of valuations, i.e. those that assign to the target formula the value t and those that assign to it the value f. If only the second class of valuations is considered, then the disjunction sign will indeed get its standard meaning (still, this is compatible with the non-categoricity of Sv –see footnote 13). However, the valuations from the first class will also satisfy the vE rule although they can be non-standard. Consequently, the Local Expression Theorem depends on the semantic assumption that the target formula of Sv is false, an assumption that cannot be inferentially justified.

It cannot be inferentially justified because a semantic property of a syntactical expression is represented (or in this context: is locally expressed) by a syntactical rule or system of rules if this expression has that property in all the models of the

[13] In the context of the Lemma from section IV above, suppose that $\boldsymbol{v}^{\vdash}$ would not satisfy the output of the rule, i.e., $\boldsymbol{v}^{\vdash}(\vdash\sigma)$ is false. This means that at least one of the inputs is not satisfied. However, since $\boldsymbol{v}^{\vdash}([\varphi]\vdash\sigma)$=t and $\boldsymbol{v}^{\vdash}([\psi]\vdash\sigma)$=t, then $\boldsymbol{v}^{\vdash}(\vdash\varphi\vee\psi)$=f. This is the case that Garson considers, when the major premise $\vdash\varphi\vee\psi$ and the conclusion $\vdash\sigma$ are not theorems. In this case, the local validity of the rule makes the disjunction sign to behave normally. Certainly, this raises no problem for $\boldsymbol{v}^{\vdash}$, since in $\boldsymbol{v}^{\vdash}$ a disjunction which is not a theorem and its both disjuncts are, indeed, simultaneously false. The non-extensionality of disjunction allows cases where the disjunction sign behaves standardly (see section V above).

rules or of the system of rules. The target formula of the vE rule is certainly not represented by any rule or calculus (since we have no reason to assume that it has the same truth value in all the models of the vE rule).

B. The possibility of a non-standard interpretation for negation and disjunction that violates the semantic rules Neg_2 and Dj_4 was investigated by Carnap (1943: 78-79; T15-8) under the assumption that the calculus preserves its soundness, i.e. no relation of logical derivability from the object language is violated. Two consequences of these assumptions were (see Carnap (1943: 79, T15-8, lit. m, n) that: **m)** every sentence that will deductively follow from both disjuncts will also follow from the disjunction itself and, thus, for soundness, will be interpreted as true and **n)** no disjunct will deductively follow from the other disjunct (otherwise it will follow from the disjunction itself and, thus, for soundness, will be interpreted as true). Consequently, if we assume that Neg_2 and Dj_4 are violated, then $\boldsymbol{v}^{\vdash}$ will be indeed an example of non-standard valuation that preserves the soundness of the propositional calculus with respect to the relation of formal derivability from the object language.

The first consequence of the assumptions is blocked by Garson through semantic stipulation (by semantically restricting the space of valuations to those that assign to the target formula of vE the value f) and, thus, should not be inferentially acceptable. The second consequence of the assumption that Neg_2 and Dj_4 are violated is blocked by Garson (2013:38) by considering the following instance of his negation introduction rule (in which B is substituted by A):

$$\frac{\begin{array}{l} A \vdash \ \ A \\ A \ \vdash \sim A \end{array}}{\vdash \sim A}$$

On the assumption that A is false, both premises are satisfied, since the antecedent is false. Thus, the conclusion has to be satisfied and, consequently, ∼A is true. However, if the relation of logical derivability is taken to be formal, if A is a closed sentence, then whatever implies both A and ∼A has to be a contradiction. Or, alternatively, if we take again the deducibility relations at their face value, i.e. if there are indeed derivations of both A and ∼A from A, then A not only has to be false, but it must have the logical form of a contradiction, case in which the conclusion is indeed a theorem. Needless to say that $\boldsymbol{v}^{\vdash}$ will again satisfy this rule since $\boldsymbol{v}^{\vdash}(\vdash\sim A)$ will be true if ∼A is a theorem.

More generally, in addition to the semantic stipulation concerning the target formula of the vE rule, there is indeed one feature of his hybrid approach to the categoricity problem that contribute in blocking $\boldsymbol{v}^{\vdash}$ (in the meta-language, as we shall see below), namely, two theorems concerning the negation and disjunction signs from the object language are disregarded by requiring the preservation of satisfaction of the natural deduction rules in a sequent format. These theorems are the following (I shall formulate them with some changes after Carnap (1943: 28, 31)):

> **T7-2.** If A and B are closed sentences, then AvB is a strongest sentence in the deductive system S with the property that if any C deductively follows both from A and B, then it deductively follows from AvB.
>
> **T8-4**. Any sentence A in the deductive system S which deductively implies both B and ~B is a contradiction.

For Carnap, a contradiction is a C-comprehensive sentence, i.e. a sentence that implies any other sentence. Let us denote with the sign '⋏' such sentences. In addition, let us use the sign '⊢' for 'deductively follows' and 'deductively implies'. Then T7-2 and T8-4 could be written as rules in a sequent format as follows:

T7-2R
$$\frac{A \vdash C \qquad B \vdash C}{A\mathrm{v}B \vdash C}$$

T8-4R
$$\frac{A \vdash B \qquad A \vdash {\sim}B}{A \vdash \curlywedge}$$

T7-2R resembles the vE rule, while T8-4R resembles Garson's (2013: 36) ~I rule (if 'A ⊢ ⋏' is substituted with '⊢~A'). However, while T7-2 and T8-4 are meant to assert some facts concerning the deducibility relation that takes place among the sentences from the object language, T7-2R and T8-4R are simply schemes of inference whose expressive power, if encoded by local models, would provide the same result concerning the standard meaning of disjunction and negation signs (from the meta-language).

With these observations, let us consider the instance of the v-Elimination rule from the objection again, with the meta-meta-variables p, q:

$$\frac{\vdash p \, \mathrm{v}{\sim}p \qquad [p] \vdash q \qquad [{\sim}p] \vdash q}{\vdash q}$$

This instance is, properly viewed, in the meta-meta-language. The horizontal line makes a conditional assertion about the relation of logical derivability (⊢) from the meta-language. On the semantical assumption that q is false and on the assumption that the horizontal line preserves sequent satisfaction, we get the result that the logical signs from the meta-language have standard meanings. But we still know nothing about the meanings of the logical symbols from the object language. On the connection between the relation of logical derivability from the object language

(/) and that from the meta-language (⊢), Garson (2013: 9) makes the following observation:

> In the case of ND systems, the symbol '/' is assumed to be in the object language, and a rule takes one from an argument or arguments to a new argument. The symbol '⊢' is used in the metalanguage to indicate the provability of an argument in a system being discussed. Therefore, 'H⊢C' abbreviates the claim that the object language argument H/C has a proof in that system.

The natural deduction rules that Garson works with make claims about the relation of logical derivability from the meta-language (⊢). The relations of logical derivability expressed in the meta-language (H⊢C) basically assert that the argument H/C has a proof in the system of logic under investigation. Thus, if 'H⊢C' is asserted, this means that the argument 'H/C' from the object language is provable. Thus, the additional proof-theoretic assumption used in the proof of the Lemma from Section III, is justified, since the premise sequents of the vE rule assert that the relations of logical derivability (/) do establish in the object language. Therefore, since each of the three arguments /Av∼A, A/C, ∼A/C has a proof, then /C will also have a proof and, consequently, $\boldsymbol{v}^{\vdash}$(C) will be true. Consequently, the disjunction sign from the object language is non-extensional, since the relation of logical derivability from the object language is compatible with $\boldsymbol{v}^{\vdash}$. The encoding of the expressive power of the sequent rules by local models does not affect the relation of formal logical derivability from the object language, language to which the syntactical sign for disjunction belongs.

Consequently, if the relations of logical derivability from the meta-rules are taken to encode formal logical reasoning, as an inferentialist should require, then logical inferentialist should thus look for other options for blocking $\boldsymbol{v}^{\vdash}$.

7. IS IT POSSIBLE TO FORMALIZE DISJUNCTION SUCH THAT $\boldsymbol{v}^{\vdash}$ AND $\boldsymbol{v}^{\vee}$ ARE BLOCKED?

As I mentioned in section II above, there are various solutions for obtaining a categorical formalization of classical propositional logic. A proper understanding of the categoricity problem reveals, however, that a purely semantical solution is, strictly speaking, an *ignoratio elenchi*, since a categorical formalization has to be obtained by using syntactical instruments. For instance, Bonnay and Westerståhl (2016: 727) block the valuation $\boldsymbol{v}^{\vdash}$ by using the requirement of the compositionality, according to which the semantic value of a complex expression is determined by the semantic values of its constituents plus the mode of composition. For the propositional logic, however, this requirement amounts to nothing more than to the requirement that the logical symbols are extensional:

(#-compositionality) For every n-ary syntactical connective # there is a semantic composition function $F_{\#}$ such that for all sentences $\varphi_1,..,\varphi_n$: $\boldsymbol{v}(\#(\varphi_1\ldots\ \varphi_n)= F_{\#}(\boldsymbol{v}(\varphi_1),\ldots,\boldsymbol{v}(\varphi_n))$.

This means that $F_{\#}$ is a truth function and if we take $F_{\#}$ to interpret #, and # is taken to be the disjunction sign, then we have that $\boldsymbol{v}(\varphi \vee \psi)= \boldsymbol{v}(\vee)(\boldsymbol{v}(\varphi),\boldsymbol{v}(\psi))$. But if $\boldsymbol{v}(\varphi)=\boldsymbol{v}(\psi)=f$, then $\boldsymbol{v}(\varphi \vee \psi)=\boldsymbol{v}(\vee)(\boldsymbol{v}(\varphi),\boldsymbol{v}(\varphi))=f$, since $\boldsymbol{v}(\vee)(f,f)=f$. Thus, $\boldsymbol{v}(\varphi \vee \psi)$ cannot be t if $\boldsymbol{v}$ is required to be compositional. Certainly, this solution blocks $\boldsymbol{v}^{\vdash}$ (and by extension $\boldsymbol{v}^{\vee}$), but this is done by semantic stipulation and not in an inferential way.

The other most promising syntactical solutions for solving the categoricity problem are to adopt a multiple conclusions formalization or a bilateralist one. I shall briefly discuss below the way in which these two kind of formalizations succeed in blocking the non-standard valuation $\boldsymbol{v}^{\vdash}$ (and by extension $\boldsymbol{v}^{\vee}$).

Carnap (1943) himself proposed a multiple conclusions formalization of propositional logic in order to fully capture the standard meanings of its logical terms. In addition to the vI rules, the disjunction operator is governed in his calculus by the following rule: ⌜AvB ⊢ A, B⌝, which may be seen as a v-Elimination rule. In the particular case in which $\boldsymbol{v}^{\vdash}(A)= \boldsymbol{v}^{\vdash}(B)=f$, the validity of this sequent requires $\boldsymbol{v}^{\vdash}(A \vee B)$ to be t and, thus, the disjunction sign gets its standard meaning. Likewise, let us consider $\boldsymbol{v}^{\vdash}$ and the following instance of the rule ⌜Av∼A ⊢ A, ∼A⌝. If $\boldsymbol{v}^{\vdash}(A \vee \sim A)=t$, then $\boldsymbol{v}^{\vdash}$ cannot assign f both to A and to ∼A, otherwise soundness would be lost. We see thus that a multiple conclusions formalization fulfils its objective. Certainly, there are various doubts for using this kind of formalization and they mainly regard the circularity of defining de disjunction operator by the mentioned rule (since the comma from the consequent may be seen to implicitly presuppose the standard meaning of disjunction) or the naturalness character of this formalization (since people do not seem to employ often a multiple conclusion rule in their ordinary reasoning), but these doubts do not represent a dead end.[14] Besides blocking $\boldsymbol{v}^{\vdash}$, another merit of this solution is that it generalizes easily for blocking the non-standard valuations for the first-order quantifiers by allowing the derivation of a universally quantified sentence from its potentially infinite denumerable number of instances (by the ω-rule) and the derivation of a potentially infinite number of instances from an existentially quantified sentence.[15]

Likewise, Carnap (1943) proposed the formulation of a rejection rule which forbids having all sentences true in a logical calculus: ⌜$V^{\&} \vdash \Lambda^{\vee}$⌝. ⌜$V^{\&}$⌝ is the universal conjunctive, which is semantically defined as being true when all sentences are true, and ⌜$\Lambda^{\vee}$⌝ is the null disjunctive, which is semantically by definition false. Thus, if we consider a valuation in which all sentences are true, then this rule becomes unsound, since this valuation will make the premise of the rule true and the conclusion false. Smiley (1996) revived this idea and Rumfitt (2000) developed it in a systematic manner by constructing a bilateral formalization of propositional logic. Rumfitt (2000) introduces two force indicators that express propositional attitudes and which are not logical operators *per se*: ⌜+⌝ (the assertion indicator) and ⌜–⌝

[14] For a discussion of these aspects see Restall (2005), Steinberger (2011), Dicher (2020).

[15] See Carnap (1943: 144-47), Shoesmith and Smiley (1978: 95-98, 366-74), Brîncuș (2024a, b).

(the denial indicator). These indicators will always prefix elementary or complex sentences, but they will not occur is the internal structure of the latter. The meaning of these indicators is such that ⌜+A⌝ will mean ⌜A? Yes⌝ and ⌜–A⌝ will mean ⌜A? No⌝. The bilateralist rules that are meant to block the non-standard valuations for the propositional logical operators are the following:

$$\text{+I}\sim \quad \frac{+(\sim A)}{-A} \qquad\qquad \text{–Iv} \quad \frac{-A \quad -B}{-(A\vee B)}$$

The first rule (+I~) leads us from the assertion of a negated sentence to the denial of that sentence, while the second rule (–Iv) leads us from the joint denial of A and of B to the denial of AvB. Before explaining how these rules of these rules block the non-standard valuations $\boldsymbol{v}^{\vdash}$ and $\boldsymbol{v}^{\vee}$, we have to define the notion of *bilateralist validity*: Γ⊢φ is valid iff for all correct valuations $\boldsymbol{v}$, if $\boldsymbol{v}(\psi)=1$ for all ψ∈Γ, then $\boldsymbol{v}(\varphi)=1$. By $\boldsymbol{v}(\psi)=1$ is meant that ψ correctly assertable in $\boldsymbol{v}$. The latter notion is defined by the following clauses (I follow here Murzi and Hjortland (2009)):

(C1) $\boldsymbol{v}(+A)=1$ iff $\boldsymbol{v}(A)=t$
(C2) $\boldsymbol{v}(-A)=1$ iff $\boldsymbol{v}(A)=f$

Clause C1 tells us that a wff prefixed with the sign ⌜+⌝ is correctly assertable under a valuation $\boldsymbol{v}$ iff the $\boldsymbol{v}$ assigns t to that wff, while a formula prefixed with the sign ⌜–⌝ is correctly assertable under a valuation $\boldsymbol{v}$ iff $\boldsymbol{v}$ assigns f to that wff.

The +I~ rule blocks a valuation that assigns t to all wffs of the propositional language since if $\boldsymbol{v}(+(\sim A))=1$, then this entails that $\boldsymbol{v}(\sim A)=t$. However, since the rule is bilaterally valid, then $\boldsymbol{v}(-A)$ has to be correctly assertable, i.e. $\boldsymbol{v}(A)=f$. Thus, A and ~A cannot both be true. Does the rule –Iv block $\boldsymbol{v}^{\vdash}$ and $\boldsymbol{v}^{\vee}$? Well, if $\boldsymbol{v}(-A)=1$ and $\boldsymbol{v}(-B)=1$, then this means that $\boldsymbol{v}(A)=\boldsymbol{v}(B)=f$. The bilateralist validity of this rule entails that $\boldsymbol{v}(-(A\vee B))=1$, which means by (C2) that $\boldsymbol{v}(A\vee B)=f$. Consequently, the valuations $\boldsymbol{v}^{\vdash}$ and $\boldsymbol{v}^{\vee}$ that assign f to the disjuncts of a disjunction and t to the disjunction itself are blocked by the soundness of the –Iv rule.

The two force indicators are introduced by Rumfitt (2000) on the basis of two structural rules (co-ordinate principles) that are very similar to the operational rules that govern the negation operator (Reductio* and Law of Non-Contradiction*):

RED*

$$\frac{\begin{array}{c}\varphi\\ \vdots\\ \bot\end{array}}{\varphi^*}$$

LNC*

$$\frac{\varphi,\ \varphi^*}{\bot}$$

where φ stands for formulas prefixed with one of the two force indicators such that φ* is the reverse of φ (i.e. if φ is +A, then φ* will be –A, and conversely). This way of introducing the two force indicators raises doubts concerning the semantical independence of the denial sign from that of the negation operator, because the two structural rules resemble very well the natural deduction rules for negation. However, since we can easily imagine a community who has force indicators for the propositional contents that are governed by these rules, but has no negation operator, these doubts could be *prima facie* set aside (see Incurvati and Smith (2010)). Although the bilateralist formalization blocks the non-standard valuations for the propositional calculi, there is no clear way in which this solution could be generalized at the quantificational level (for instance, Warren (2020) uses the bilateralist approach at the level of propositional operators, but embraces open-endedness for dealing with the quantifiers). From this perspective at least, the multiple conclusions formalizations seems to be a better option if one wants a unified approach.

A more recent unified approach for solving the categoricity problem for propositional, first- and second-order logics has been proposed by Murzi and Topey (2021). An essential ingredient of their approach is the use of Garson's local models for the natural deductive systems associated with these logics. In the case of propositional logic, they take for granted Garson's Local Expression Theorem and embed the negation sign at the structural level for escaping the incompleteness feature of the natural deduction rules for the material implication (Pierce's law cannot be derived only with the rules for the material implication[16]). However, although they escape the incompleteness phenomenon, due to the reasoning conducted in section IV, their use of the local models is still insufficient for uniquely determining the meaning of disjunction since the valuations $\boldsymbol{v}^{\vdash}$ and $\boldsymbol{v}^{\vee}$ discussed above are members of a local model for the system Sv or for an extension of it.

7. FINAL REMARKS

Although the local models of the natural deduction systems for classical propositional logic do not provide sufficient reasons for reading off a unique interpretation of the logical symbols that its rules introduce, if the deducibility relation preserves its formal character, a categorical propositional calculus can be obtained if alternative formats of the logical propositional calculus are adopted. The multiple conclusions and the bilateralist formalizations are good options for eliminating the non-standard valuations $\boldsymbol{v}^{\vdash}$ and $\boldsymbol{v}^{\vee}$. These systems are not entirely free of (semantical) philosophical assumptions, but the logical inferentialists can decide, on the long run, which formalization is less problematic from an inferentialist point of view.

[16] This is actually one of the reasons for which Garson (2013) favours the global models to the local ones in encoding the expressive power of the natural deduction rules for classical logic.

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