ON SUTHERLAND'S VIEW REGARDING KANT'S PHILOSOPHY OF MATHEMATICS

MARIAN-VALENTIN POPESCU

Abstract. Sutherland's main thesis is based on the idea that, in the exegesis on Kant's mathematics, the role of intuition was rather ignored. He holds that the role of intuition is related to what would represent the foundation of Kant's philosophy of mathematics, namely, the "Eudoxian theory of proportion" presented in Euclid's *Elements*. Thus, Sutherland tries to bring geometry, arithmetic and algebra under the same umbrella of the Kantian philosophy of mathematics. Starting from one of Sutherland's studies, our interest is directed not necessarily to a criticism of his interpretation, with which we generally agree, but rather to reframe it in the context of Kantian syntheses. By formulating, in the final part, an exemplary case of algebra, I will try to illustrate the validity of the unity between algebra and geometry, as well as how the character of Kant's synthetic a priori may be address here. Therefore, I do not intend a criticism of Sutherland's attempt to unify the three fields of mathematics, but rather its reconsideration in order to show that the foundation of Kant's mathematics is, in fact, the *figurative synthesis* involved in pure a priori intuition.

Keywords: Kant; mathematics; Sutherland; theory of proportions; Euclid.

Kant's philosophy of mathematics became an almost separate field of exegetical-systematic research, if we look at the "tradition" of readings of this type from the German philosopher, especially from the last century. As Daniel Sutherland observes, in the last 40–50 years, attention to the identity, place and role of mathematics in transcendental philosophy was initially stimulated especially by the works of Jaakko Hintikka and Charles Parsons; I should add here the "tradition" originated in Frege's criticism on Kant especially from *Foundations of Arithmetic* (1884) and, nowadays, by the recent dispute on "Kant–Frege view".

Regarding the relationship between arithmetic (algebra) and geometry in Kant's mathematics, some relatively recent researches have opened a new perspective, or,

Marian-Valentin Popescu ✉
Mathematics and Computer Science Research Center, Technical University of Civil Engineering, Bucharest, Romania

Rev. Roum. Philosophie, **66,** *2*, pp. 339–354, București, 2022

more precisely, have reinforced it[1]. For our theme and purposes, however, we took into account especially Sutherland's provocative view developed in his research *Kant on Arithmetic, Algebra, and the Theory of Proportions*[2]. Sutherland's main thesis lays on the idea that, in the specialized literature regarding Kant's mathematics, the role of intuition was rather ignored. In the interpretation he proposes regarding Kant's mathematics in general, the role of intuition is linked to what would represent the foundation of Kant's philosophy of mathematics, especially in the *Critique of Pure Reason* (A/B)[3], namely, the "Eudoxian theory of proportion" presented in Euclid's *Elements*. The author thus tries to bring geometry, arithmetic and algebra under the same umbrella of the Kantian philosophy of mathematics.

Starting from Sutherland's research, our interest is directed not necessarily to a criticism of his interpretation, with which we generally agree, but rather to reframe it in a wider Kantian context; in the final part we will also try to exemplify some of Sutherland's conclusions through an exemplary case of algebra, trying to illustrate the validity of the unity between algebra and geometry, as well as how the character of Kant's synthetic a priori may be address. We therefore do not intend a criticism of Sutherland's attempt to unify the three fields of mathematics, but rather its reconsideration in order to show that the foundation of Kant's mathematics is, in fact, the figurative synthesis involved in pure a priori intuition.

I will present below, synthetically, the context addressed by Sutherland as well as the support of his most courageous assumptions. First of all, he complains about the neglect, in more recent exegesis, of the importance of Kant's theory of magnitude and the role of intuition in the Kantian philosophy of mathematics. With the intention of unifying Kant's conception of arithmetic, algebra and geometry, Sutherland sustains that Kant's theory of magnitudes is related to and depending of the Eudoxian theory of proportions. This task is required not only in order to be able to respond with a unifying solution to divergent and (apparently) irreconcilable problems of how Kant's conception of mathematics has been understood, but also Sutherland argues in order to understand Kant's view on human cognition in general[4].

Sutherland's first assumption is that, given that Eudoxius' theory of proportions provides a mathematical treatment of continuous magnitudes, it could be directly related to Kant's geometry. One of the problems that he points out is the unclear connection be-

[1] I'm reffering here to Daniel Sutherland's researches between 2004–2008: "The Role of Magnitude in Kant's Critical Philosophy", *Canadian Journal of Philosophy*, vol. 34, nr. 3, 2004, pp. 411–442; "Kant's Philosophy of Mathematics and the Greek Mathematical Tradition", *The Philosophical Review*, vol. 113, nr. 2, 2004, pp. 157–201; "The Point of Kant's Axioms of Intuition", *Pacific Philosophical Quarterly*, vol. 86, nr. 1, 2005, pp. 135–159; "Kant on Arithmetic, Algebra, and the Theory of Proportions", *Journal of the History of Philosophy*, vol. 44, nr. 4, 2006, pp. 533–558; *Arithmetic from Kant to Frege: Numbers, Pure Units, and the Limits of Conceptual Representation*, Cambridge, Cambridge University Press, 2008.

[2] Daniel Sutherland, "Kant on Arithmetic, Algebra, and the Theory of Proportions", pp. 533–558.

[3] The references on Kant's first *Critique* will be from Imm. Kant, *Critique of Pure Reason*, translated and edited by Paul Guyer and Alan Wood, Cambridge, Cambridge University Press, 1998.

[4] D. Sutherland, *ibidem*, p. 558.

tween the understanding of mathematics in general as an extension of the way in which geometry is approached (as a science of quantities of the *continuum* type) with the way Kant relates to *discrete* mathematical entities, found in his arithmetic. The major contribution of Sutherland's research would therefore focus on the development of a perspective that would make the Kantian view of mathematics in general more comprehensible and plausible by reconsidering arithmetic in the Kantian theory of magnitude. Such a reconsideration would make the same foundation that stands for geometry (i.e. Eudoxian theory of proportions) common to Kant's arithmetic and algebra (algebra would be a generalization of Kant's arithmetic). In other words, the whole of mathematics could have as its foundation the Eudoxian theory of the proportions of continuous quantities. Sutherland sees three problems with which his perspective should be accommodated, which presuppose the difficulty of reducing arithmetic to geometry, and this fact requires a closer look at the relationship of arithmetic with algebra in Kant.

With respect to the Greek tradition on number and the theory of proportions, Sutherland accounts for the Platonic understanding of numbers with its two meanings: one related to the number of actual, concrete things in a descriptive sense, as description of a collection of objects of possibly different and countable qualities, and the other abstract, referring to "pure" numbers, which can only be thought. Unlike the first meaning, the latter excludes any differentiation between the respective units, each unit being identical to any other; in Plato, these pure numbers have an independent existence, that Aristotle rejects[5]; also, pure units are indivisible, from which it follows that only whole numbers are, strictly speaking, numbers. This second meaning, of pure numbers, seems to be the one that would have influenced Kant as well.

We now reproduce, briefly, Eudoxus' theory of proportions, as it was synthesized by Sutherland. The definition of ratios is that of Euclid's *Elements* ("Book five")[6], where ration is defined as "a sort of relation with respect to size between two homogeneous magnitudes". Sutherland formulates this definition of ratio using the "property of Archimedes" in a notation of modern algebra which assumes the following relations on 4 quantities a, b, c, and d in certain ratios of multiplication by two scalars, m and n: $a:b = c:d$ iff for all m, n: $ma > nb \rightarrow mc > nd$; $ma = nb \rightarrow mc = nd$; $ma < nb \rightarrow mc < nd$[7].

In other words, a pair of sizes is in the same relation with another pair if and only if the comparative size relation of the first pair (greater, equal, or less) is the same as the comparative size relation of the second pair under all equimultiple transformations[8]. Very important to note here is that relations and operations such as comparative size and multiplication are on these sizes as such, so they are not on the numbers that would correspond to the sizes of these sizes. The precious consequence here is that in the Eudoxian theory of proportions numbers are not encountered and used to express ratios, which equates to the possibility of being applied including to *incommensurable* quantities. This

[5] Cf. *ibidem*, p. 535.
[6] Euclid, *Elements*, cited in *ibidem*, p. 536.
[7] *Ibidem*, p. 536.
[8] *Ibidem*.

new perspective was taken up by Euclid, and under this development, both numbers and incommensurable quantities can be in ratios; or, both *continuous* and *discrete* quantities (numbers understood as quantities) can be subject to be written in the form of ratios[9].

This tradition inaugurated by Euclid, therefore, focuses more on the continuous quantities of geometry rather than on arithmetic ones, developing a mathematics that makes use as little as possible of numbers; it is Sutherland's idea mentioned above, that of the priority of geometry over arithmetic, which would have had a decisive influence on thinking about mathematics not only in the 18th century in general, but especially in Kant.

Sutherland then discusses the important influences that the theory of proportions would have had on Kant's philosophy of mathematics: among other things,he considers the concept of "homogeneous size". For Euclid, this type of size is defined, in the simplest way, as that size which is made up of sizes that can be in a relationship (for example: lines, surfaces/areas, numbers); Euclid's strong condition for the admission of homogeneous quantities is that each quantity, in order to be in a relationship with another, must be capable of exceeding the other when multiplied. From this it follows that the quantities in a ratio in which one of them presupposes either the infinitesimal or the infinite are exceptions[10].

In essence, the Eudoxian theory of proportions is based on the composition of magnitudes and on the comparable magnitude relations of equality, of the greater and of the lesser, respectively; or, more clearly, on the elementary relationships of composition, *part-whole* and *equality*[11]. Therefore, Sutherland argues that the theory of proportions expresses the basic mathematical properties of quantities and takes it into account with respect to the way the composition relationship, *part-whole* and *equality*[12] relationships of homogeneous quantities are understood and structured in philosophy Kant's geometry; moreover, he argues that these elements determine even the philosophy of arithmetic and algebra of the German philosopher.

I will show very briefly the way Sutherland understands mathematical cognition in Kant. He starts with the presentation of the theory of magnitude, which would be the basis of Kant's philosophy of mathematics in general, and the latter would be based on the Eudoxian theory of proportions – this last assumption represents the main thesis of Sutherland's research. Claiming the lack of a section dedicated to explaining the possibility of mathematical knowledge in Kant's work, he reveals the difficulty in general of a performing interpretation on this subject.

Kant understands the concept of homogeneity or size as proportions of comparable sizes. Sutherland's proposal would be that the part of the *Analytics* dedicated to the Axioms of Intuition[13] be considered one of the most relevant sections where Kant dis-

[9] *Ibidem*, p. 537
[10] *Ibidem*, p. 537 and note 15.
[11] *Ibidem*, p. 538.
[12] *Ibidem*, p. 538.
[13] Imm. Kant, *Critique of Pure Reason* (from now on, I will quote simple *Critique*), B 202–B 207/A 166.

cusses the nature of the cognition of mathematics and consequently its very main starting point. Indeed, here Kant explicitly says that mathematical cognition is cognition of quantities. For Kant's view on sizes is derived from the Euclidean tradition, Sutherland believes that the Kantian approach to mathematical cognition involves the type of cognition that makes possible the very theory of proportions under discussion. The regime under which these cognitions stand in Kantian thinking includes the cognitions about comparable sizes, which can be reduced to the cognitions of *equality* and *part-whole* type relations, found in Eudoxian theory; also, these cognitions presuppose the cognition of the *equality* and *part-wholecomposition* relations of the quantities.

As for mathematical *homogeneity*, Kant actually defines quantity as the homogeneous multiple (manifold) in intuition, which, Sutherland considers, reflects the Eudoxian-Euclidean conception of homogeneous quantities as it appears in Book 5 of Euclid's *Elements*. It is about the composition in which the parts of the same kind can be composed to obtain something bigger of exactly the same type – it is about elements such as lines or surfaces, planes or volumes, which are homogeneous with each other (lines with lines, planes with planes etc.) and to which we refer as "mathematical homogeneities"[14].

Next, Sutherland claims that Imm. Kant would focus on the composition of homogeneous mathematical quantities in his attempt to give an explanation to mathematical cognition. The condition of such knowledge is the representation of numerical, quantitative differences, without qualitative differences; considering the multiple of the numerical difference without qualitative difference distinguishes between quantity and quality. For more clarity, Sutherland calls the notion of homogeneity, which expresses a condition of cognition that does not allow the representation of any qualitative difference, "strict homogeneity".

Summarizing, the "mathematical homogeneity" assumed by the Eudoxian theory of proportions is characterized by the representation of composition and part-whole relationships, respectively equality. For Kant, this means that representing these types of relations about the homogeneous multiple allows us to have access to the homogeneity of mathematics, that is, to the mathematical relations of quantities. Next, Sutherland uses the concept of strict homogeneity characterized by him above in relation to the mathematical character of quantities in Kant by relating the Kantian concept of space determined as an extensive quantity with spaces determined as wholes made up of parts; this is how the possibility of representing the part-whole relations of a spatial quantity and those between the spatial quantities is justified, and this allows us to recognize them as comparative size relations.

Perhaps the most difficult task assumed by Sutherland is, now, that of being able to "reconcile" what Kant would have first had in mind as continuous quantities (in particular the continuous quantities of geometry) with what arithmetic implies on this level: numbers and discrete sizes, or collections of discrete or even distinct things. It is actually the task of trying to assimilate arithmetic to geometry.

[14] *Ibidem*, p. 539.

The *synthetic* character of arithmetic is brought into discussion by the fact that arithmetic tells us more than we can assume from the analysis of simple concepts, exclusively through the intellect. Kant's arithmetic was most often linked to his claim that, in order to be able to represent particular numbers, *intuition* is necessary.

Sutherland's first conclusion here is that the type of connection between arithmetic (understood by these features) and the general theory of quantities is one of incorporating the former into the latter; the author offers as an example what Kant says at B 15–16 of the Introduction, a passage that we reproduce in full below:

> To be sure, one might initially think that the proposition "7 + 5 = 12" is a merely analytical proposition that follows from the concept of a sum of seven and five in accordance with the principle of contradiction. Yet if one considers it more closely, one finds that the concept of the sum of 7 and 5 contains nothing more than the unification of both numbers in a single one, through which it is not at all thought what this single number is which comprehends the two of them. The concept of twelve is by no means already thought merely by my thinking of that unification of seven and five, and no matter how long I analyze my concept of such a possible sum I will still not find twelve in it. One must go beyond these concepts, seeking assistance in the intuition that corresponds to one of the two, one's five fingers, say, or (as in Segner's arithmetic) five points, and one after another add the units of the five given in the intuition to the concept of seven. For I take first the number 7, and, as I take the fingers of my hand as an intuition for assistance with the concept of 5, to that image of mine I now add the units that I have previously taken together in order to constitute the number 5 one after another to the number 7, and thus see the number 12 arise. That 7 should be added to 5 I have, to be sure, thought in the concept of a sum = 7 + 5, but not that this sum is equal to the number 12. (B 15–16).

Therefore, Sutherland discusses the passages from B 15–16, where Kant implicitly speaks about the synthetic character of arithmetic in that, no matter how much we analyze the simple concept of sum or unification of numbers, we do not obtain the sum if we do not take *intuition* as help (for example, analyzing the concept of the sum between 7 and 5 does not automatically lead us to the number 12). Kant's conclusion would be that, no matter how much we *think* about the unification of two numbers, unless we consider composition in intuition we cannot determine what the sum is and what exactly it is equal to.

Sutherland holds that Kant had in mind the intuitive synthesis that underlies composition especially in the *Schematism*[15], where he says that number "is a representation that sums up the successive addition of one by one" [B 182]. Then Sutherland acknowledges that *composition* is of central importance in Kant: he identifies it with *figurative synthesis* [B 151], which operates when we draw a line. Sutherland calls this a *synthesis of composition* or *composition's synthesis*. At the base of the *composition* is therefore the figurative synthesis, the one that underlines the mathematical synthetic principles [B 201 n.]. The Kantian synthesis of the composition would correspond to

[15] Imm. Kant, *Critique*, A 137/B 176–A 147/B 187.

the special composition whose homogeneous quantities and only they can be its object or can be operable here; the representation of this composition is the one required *in intuition*, both in arithmetic and in the generation of continuous quantities of geometry.

This connection between number and arithmetic composition is also present in other passages in Kant, where it requires that "added particular numbers be equal to their total particular", which would imply the appeal to part-whole relationships. Sutherland then discusses the concept of number, that in Deduction A is related and even belongs to the category of totality (&11), and argues that it must be seen as a whole [A 99]. The number requires not so much the multiplicity of parts but its cognition as a whole, which is possible through the category of totality. [B 111]. These elements, Sutherland believes, are suggestive of Kant's larger theory of magnitude.

The question that arises is how the discrete quantities of arithmetic are related to the continuous quantities of the theory of proportions? Sutherland tries to explain that the discrete quantities of arithmetic are related and assimilated to the continuous quantities of the theory of proportions by the fact that Kant's theory of arithmetic rests on the theory of proportions, which makes his perspective on numbers and discrete quantities assimilated to the perspective on continuous quantities. Sutherland relates the special attention that Kant pays to geometry and geometric examples in his work with the priority that the "continuous quantities" have in the corresponding passages of the "Axioms of Intuition". He also says that, in a *Lecture of Metaphysics* (29:979, pp. 1794–5), Kant would argue that the homogeneous multiple of space and time allows us to capture the *part-whole* size relationships. Here we are referring to the category of quantity, not to the size as such. Kant's perspective on arithmetic as well as on geometry is based on the theory of proportions, and our knowledge of the truths of arithmetic rests on the same cognitions that underlie this theory.

Then Sutherland focuses on the definition of number in Euclid, where "number is a collection of units"; consequently we have the theory of proportions and of proportions between *numbers* which actually replicates the Eudoxian theory of proportions. So numbers would just be a special case of magnitudes[16]. Sutherland sees the assimilation of numbers in Kant's perspective on continuous quantities through the way Aristotle understood the relationship between the measurable and the measurement process, which assumes "unit", "one" as a unit of measure. Aristotle distinguishes between measure (1, unit) and the measurable. Each number is said to be more than one because the number is made up of units and because each number is measurable by one, by unit[17]. In Aristotle it seems that "one", "unity" is the measure of number. From this point of view, the counting of discrete quantities can be regarded as a kind of measurement. Therefore, in counting, we specify the unit of measure and proceed to the addition operation progressively towards the required totality. Euclid suggests a similar interpretation – of counting as

[16] See note 29 of D. Sutherland, *ibidem*.

[17] Here we note the assimilation of this perspective to the set represented by what, with modern algebra, is called the trivial group of cyclic groups. In our discussion, the emphasis falls not only on the generator unit, but also on the measurement process.

measurement. Sutherland argues that this approach could provide a unifying perspective on considering discrete and continuous quantities together.

The validity of Sutherland's claim on assimilating counting to the measurement of continuous quantities presupposes to think of numbers as represented by dimensions/lengths. The concepts of number can emerge from line-generating synthesis when line lengths are marked by a unit. This provides Kant with a natural way to assimilate arithmetical knowledge in the perspective of the cognition of continuous quantities, and hence, in the theory of proportions.

As for Kant, he sees a close relationship between arithmetic and algebra; speaking of arithmetic, he claims that algebra uses general signs for numbers. In the *Critique*, Kant distinguishes two kinds of quantities: *quantitas* and *quanta*. "Arithmetical formulae" and algebra are grouped together under the rubric quantitas, in contrast to the quanta of geometry. (B 204/205; A 717/B745, p. 17) As Wallis, Sutherland also claims that Kant would have thought of algebra as universal mathematics, in particular, that it would express the general doctrine about quantities[18].

Sutherland then lists passages from Kant where he claims the opposite of what he wants to prove, that algebra is the general doctrine of quantities. These passages suggest that algebra either has no object, or at most we are given the concept of a thing in general. In the *Discipline of Pure Reason*, for example, Kant claims that in algebra, mathematics completely abstracts from the nature of the object that must be thought according to such a concept of size [A 717/B 745]. Sutherland claims that in fact the object of algebra is quantity as a pure quantity, and not that it hasno object at all (which is what we also claim). This view is reinforced by the fact that algebra was conceived as a technique or method for solving problems rather than as a discipline with its own domain of objects. He believes that algebra considers quantities without qualities, that is, that it has as its object *pure* quantitative properties, and quantity is pure because it is the object of all mathematics. Algebra abstracts from qualities, but not from pure quantitative properties, that is, from their character of magnitudes. Sutherland then refers to the passage in Kant:

> But mathematics does not merely construct magnitudes (quanta), as in geometry, but also mere magnitude (quantitatem), as in algebra, a where it entirely abstracts from the constitution of the object that is to be thought in accordance with such a concept of magnitudes. [A 717/B 745].

Thus, if Kant claims that algebra is the general doctrine of quantities and if Kant's theory of quantities rests on the Eudoxian theory of proportions, then the Kantian understanding of algebra, i.e. algebra, will also rest on this theory of proportions.

Furthermore, Sutherland shows the connection between geometry and algebra by recourse to the history of mathematics, starting with what Euclid states in proposition 4 of the 5th and 6th books as relations between quantities that correspond to algebraic statements – it is presented the translation into the algebraic formula $(a+b)^2=a^2+b^2+2ab$ of the corresponding geometric problem[19]. Sutherland concludes that the theory of proportions

[18] D. Sutherland, *ibidem*, p. 549.
[19] See note 56, *ibidem*.

in books 5 and 6 of Euclid is *geometric* algebra, particularly because it is applied by Euclid to geometric relations, but the theory as such concerns *all quantities*. Also, many important algebraists of that early modern period emphasized the connection between the Diophantine algebra of numbers and the Greek theory of proportions. Algebra involves equations in solving problems. Viète linked this conception of algebra with the Eudoxian theory when he established that "a proportion can be called the construction of an equation, and an equation the solution of a proportion". Viète was inspired in his achievements and draws his creative mathematical inspiration from the Eudoxian theory of proportions as it appears in Euclid, more precisely when he formulated his algebra[20]. Viète's most suggestive achievement for Sutherland's purposes is an algebraic method common both to the calculation of numbers and to the Eudoxian theory of the proportions of magnitudes. Descartes, Wallis and Wolff are also mentioned.

The first challenge that Sutherland tries to address concerns the apparently irreducible differences, he says, of the syntheses corresponding to arithmetic and geometry respectively: the first one would be a special synthesis, and that of geometry a more general one (or a synthesis of composition emphasizing arithmetic, and a general one emphasizing geometry). The synthesis of composition in general and of geometry, of continuous quantities, is the synthesis of the homogeneous or the synthesis of "part to part", and the synthesis of the composition of arithmetic, or of *discrete* quantities, is that of "units"[21]. Indeed, Kant does not use lines to represent numbers, but fingers. Although Segner quoted by Kant also refers to lines that represent numbers, Kant does not use this reference (as well as in other places in the *Critique*, where Kant always refers to discrete and separate, isolated units). These passages rather suggest that here Kant reflects the Greek conception of number rather than the modern one: Kant allows rational numbers, which can be expressed as a ratio of two numbers made up of units. As he says in a letter to Rehberg[22] from 1790 regarding the representation of irrational quantities, he does not admit irrational numbers. Sutherland then claims that, by appealing to discrete quantities such as fingers and points, Kant believes that the synthesis of composition underlying arithmetic is the synthesis of discrete and discontinuous units. Then, explicitly, Kant says in the *Critique* that at the base of every number there must be unity. Kant insists that there is a type of synthesis by which we can represent the composition of discrete and discontinuous units in a collection. Sutherland believes that this arithmetic synthesis is different from the synthesis that generates the representation of a continuous quantity in the drawing of a line, or that it is the same underlying synthesis exercised in a different manner. Furthermore, the categories of quantity provide a mereological foundation for mathematics in two different ways, for they can be involved in knowing the part-whole relations of discrete collections as well as in knowing the relations of continuous quantities. For this reason, Sutherland argues, Kant does not intend to assimilate arithmetic directly into the theory of proportions by thinking of numbers as represented by lines or by understanding counting as the measurement of units of size. Sutherland also says that Kant refers to the

[20] Sutherland, *ibidem*, p. 550.
[21] For the references of Kant's passages on this subject, see *ibidem*, p. 554 and notes 76, 77.
[22] Cf. Sutherland, *ibidem*.

arithmetical tradition in his readings of mathematics because, like others in the early modern period, he thinks of arithmetic as being about collections of discrete units, which in the appropriate paradigm are distinct in the way that dots on a page are. This view is compatible with representing numbers as *discretely known line lengths* as long as we keep in mind that there are ratios between lines for which there is no corresponding ratio between numbers. Sutherland's solution, whereby the above difference does not undermine the assimilation of the numbers of dimensions, is not supported by the Kantian text, being a proposal that the author we are dealing with makes.

The second problem is that of the interpretation of the relations between arithmetic and geometry through the Eudoxian theory of proportions and takes into account the fact that Kant's references to the problem of arithmetic number are aimed exclusively at arithmeticians (frequented by Kant between 1762 and 1764) who are anti – Euclidean by distinguishing as mathematicians from the theory of proportions and the geometric tradition that contains it. This issue is not relevant in the economics of our theme and, we believe, is derived from the above issue, being rather an exegetical matter of "historical reference"; systematically, Sutherland answered this question by addressing the first and the last problem[23].

Finally, the third problem concerns "*homogeneity*" or the distinction between types of homogeneity: we have, on the one hand, the homogeneity of a simple count and what Sutherland understands by "strict homogeneity", i.e. the one admitted by Kant in mathematics. The homogeneity of a simple fall under a common counting-concept is not everywhere as strict as a mere numerical difference, without any qualitative difference. The homogeneity of counting hardly requires intuition, Sutherland claims. Then why intuition is necessary for combinatorics that supports arithmetic, he also asks. If we are dealing with two types of homogeneity, and only the one he defines as "strict" may be considered in mathematics, then the reason why Kant appeals to intuition is also related to the Eudoxian theory of proportions.

Sutherland proposes understanding numbers through the concepts of size or length. In this way is explained the possibility that arithmetic uses strict homogeneity, the key to Sutherland's interpretation. Without strict homogeneity, we will not be able to know the mathematical character of arithmetical composition; we will not even be able to obtain the arithmetic concepts, which appear by successively putting together strictly homogeneous units. Kant's qualitatively identical units are assimilable to only numerical differences, present in mathematics. They are none other than the pure Aristotelian units that stand as pure units in all mathematics. From this standpoint, both arithmetic and geometry assume strict homogeneity, and thus intuition is necessary. Thus, arithmetic, like the theory of proportions for continuous quantities, requires or demands strict homogeneity, and hence intuition[24].

We saw above that, in Sutherland's interpretation, the foundation of mathematics resides in the Eudoxian theory of proportions. From here we have the relations of composition, *part-whole* and of *equality* as well as, in particular, the operations of addition

[23] For further details, see D. Sutherland, *ibidem*, p. 555.
[24] *Ibidem*, p. 557.

and subtraction (recall that, in mathematics, addition is often written as multiplication and subtraction as addition). Also it is useful for us his extended analysis, synthesized above, on the concepts of number, (strict) homogeneity or Kantian *a priori* syntheses. We will discuss these as well as the concept of *order* and *transcendental content* in what follows. My approach is slightly different from Sutherland's as it rests on a broader Kantian context, that of the syntheses. To present it, we now quote what Kant says about the fundamental operations of arithmetic (addition and subtraction) in his letter to Johann Schultz of 1788, a year after the publication of B edition of the *Critique*:

> I can form a concept of one and the same magnitude by means of several different kinds of composition and separation, (notice, however, that both addition and subtraction are syntheses). Objectively, the concept I form is indeed identical (as in every equation) (10:555)[25]

Kant clearly states here that the operations of addition and subtraction are *syntheses*. If we also take into account the note from B edition of the *Critique*, we may consider that these two fundamental operations, intrinsic to the relations in the theory of proportions, are based on *figurative synthesis*:

> This **synthesis** of the manifold of sensible intuition, which is possible and necessary a priori, can be called **figurative** (*synthesis speciosa*), as distinct from that which would be thought in the mere category in regard to the manifold of an intuition in general, and which is called combination of the understanding (*synthesis intellectualis*); both are **transcendental**, not merely because they themselves proceed *a priori* but also because they ground the possibility of other cognition *a priori*.[26] [B 151]

For the synthesis of succession as the addition of "one by one" is a synthesis of sensibility, and the fundamental operations of arithmetic imply this synthesis of succession, these operations are based on this synthesis. Kant distinguishes it from the one that would be thought in relation to the manifold of an intuition *in general*, in the mere category (*synthesis intellectualis*). The first, figurative synthesis, moreover, relates not only to the originally synthetic unity of apperception (to this transcendental unity), which is thought in category, but also to sensible intuition (founded in the pure intuitions of space and time) so that Kant calls it *the transcendental synthesis of imagination*. Beyond the exegetical problem, i.e., if the imagination belongs to the intellect or to sensibility, which we do not discuss here, Kant states at B 152 that, at the same time, imagination is an effect of the intellect on sensibility and represents its first application to objects of intuition possible to us:

> Yet the figurative synthesis, if it pertains merely to the original synthetic unity of apperception, i.e., this transcendental unity, which is thought in the categories, must be called, as distinct from the merely intellectual combination, the transcendental synthesis of the imagination. (B 151/152)

[25] Imm. Kant, "Letter to Johann Schultz", in *Correspondence*,*The Cambridge Edition of the Works of Immanuel Kant*, Cambridge, Cambridge University Press, 1999, p. 283.

[26] Imm. Kant, *Critique*, p. 150.

> Imagination is the faculty for representing an object even without its presence in intuition. Now since all of our intuition is sensible, the imagination, on account of the subjective condition under which alone it can give a corresponding intuition to the concepts of understanding, belongs to sensibility; but insofar as its synthesis is still an exercise of spontaneity, which is determining and not, like sense, merely determinable, and can thus determine the form of sense a priori in ac cordance with the unity of apperception, the imagination is to this extent a faculty for determining the sensibility a priori, and its synthesis of intuitions, in accordance with the categories, must be the transcendental synthesis of the imagination, which is an effect of the understanding on sensibility and its first application (and at the same time the ground of all others) to objects of the intuition that is possible for us. As figurative, it is distinct from the intellectual synthesis without any imagination merely through the understanding. [B 152].

Therefore, the understanding of Kant's *a priori* syntheses is made under two main specifications (purely intellectual and figurative); as originating in the intellect, both allow a unitary view of them, as long as both presuppose the contribution of the original synthetic unity of transcendental apperception; the differentiation occurs when we have the restriction at the level of intuition in general, where the figurative synthesis of imagination presupposes, as in mathematics (algebra and geometry), the construction in pure sensible intuitions. Kant explicitly says that this last synthesis is the expression of the originally synthetic unity of transcendental apperception, which "transports", as we will see, the transcendental content through categories at the level of pure sensible intuition. At the same time, however, as a productive imagination, figurative synthesis is also the basis of the pure intuition of space. We want to emphasize now that the same figurative synthesis is at the basis of arithmetic (primarily as a synthesis of succession in time) and at the basis of geometry (primarily as a synthesis of productive imagination[27]). From this point of view, we can distinguish two "steps" of the figurative synthesis: the first, which is assimilable to the internal sense (time) and primarily involves the sequence by determining time itself, and the second in which the productive imagination has the main role, and which is to be found also at the level of constructions in science of geometry.

Even if the status of *irrationals* would still remain undecided, the fact that, nevertheless, Kant admits *rationals* as quantities made up of whole numbers allows the development of the relations of the theory of proportions also at the level of generalized arithmetic (algebra). One of the fundamental relations here, which explains, in part, the connection between the originally synthetic unity of apperception, figurative synthesis, categories and, hence, the determination of the pure forms of sensibility (space and time) by *schematism*, is the concept of order and/or "transcendental content", which I just mentioned above:

> The same function that gives unity to the different representations in a judgment also gives unity to the mere synthesis of different representations in an intuition, which,

[27] Approximately the same position is present in Michael Friedman, but he is referring to the "productive synthesis". For a better account regarding Kant's synthesis, see Michael Friedman, *Kant and the Exact Sciences*, Cambridge, Harvard University Press, 1998, p. 76, especially chapter 1 (Geometry) and 2 (Concepts and Intuitions in the Mathematical Sciences).

> expressed generally, is called the pure concept of understanding. The same understanding, therefore, and indeed by means of the very same actions through which it brings the logical form of a judgment into concepts by means of the analytical unity, also brings a transcendental content into its representations by means of the synthetic unity of the manifold in intuition in general, on account of which they are called pure concepts of the understanding that pertain to objects *a priori*;[28] [A 79/B 105]

As a function of the intellect, through acts of transcendental syntheses (*intellectual* and *figurative*), the pure concept gives unity to different representations in a judgment, respectively a transcendental content to the simple synthesis of representations in an intuition. The concept of unity and that of "transcendental content" are the prerogative of the intellect, as a function at two levels: of logic and of intuition (in general). At the level of logic, through the actions of purely intellectual synthesis, through unity (as an analytical unit) it brings the logical form of judgment (order and coordination in judgment) into concepts, and through the actions of the synthetic unity of the manifold in intuition in general, the intellect introduces to the level of its representations a transcendental content.

From this point of view, the transcendental content at the level of representations of the intellect requires a more special type of order, better expressed by the concept of "lawfulness". In a more general sense, lawfulness can be expressed through the structure of the Kantian categorical system itself, where the 12 categories, ordered under the 4 titles, assume a certain *order* and functioning, which reflect the functional exercise of the intellect in general, and, through *schematism*, at the level of sensibility. This lawfulness is best exemplified in correlation with the concept of order when Kant discusses order in the synthesis of succession and the *a priori* character of the law of causality [eg A 192–A 194/B 237–239] – we have here the reflection of the determinative structure of the categorical system at the level of sensibility through the category of causality [A 201/B 246]. Also, as we saw in Sutherland's interpretation, the categories of quantity act at the level of arithmetic (algebra) and geometry through the concept of number as a scheme, respectively through the relations of the Eudoxian theory of proportions.

As far as *composition* relations are concerned, they seem relatively easy to put in correspondence with the two two-time manifestations of figurative synthesis: the elementary operations (of equality and addition and subtraction, etc.) of arithmetic (algebra), understood as expressing the ratios in the theory of Eudoxian proportions correspond to acts of figurative synthesis as a synthesis of the sequence of determining time itself (the reduction of algebra to geometry, Sutherland); the more complex operations of algebra and geometry (where we have not only the fundamental operations, but, in principle, any law of composition) presuppose figurative synthesis primarily through the exercise of productive imagination (synthesis of succession being presupposed in any relation to the object of the categories).

I have tried above to schematically reproduce the a priori synthetic character of mathematics in Kant by interpreting the unification of arithmetic (algebra) with geometry, achieved by Sutherland, reconsidering the determinative "route" of the originally synthetic

[28] *Ibidem*, pp. 150–151.

unity of apperception from intellect to sensibility in the construction of mathematics.

In this final part of the article I will propose an integration of the most important elements discussed (respectively the aspects established by Sutherland in the Kantian scheme expanded by the elements considered by us above) in an exemplary case of algebra: the transition from simple arithmetic operations to isomorphism of groups. Thus, I will show how Sutherland's interpretation can work, by adding those elements of Kantian philosophy discussed above in such a way as to illustrate the validity of the unity between algebra and geometry as well as the character of the synthetic a priori in Kant's philosophy of mathematics.

Most of the time in mathematics we talk about *operations* on certain sets, and here we can have in mind the Kantian syntheses[29] that stand for the elementary operations of arithmetic – addition, multiplication and subtraction, as we have shown. A simple example is the subtraction operation of rational numbers, which in our sketch perfectly assimilates elements of the theory of proportions such as the notion of "ratios". This example can be seen as a procedure by which we associate to the couple of numbers $(x, y) \in \mathbb{Q} \times \mathbb{Q}$ the rational number $x - y \in \mathbb{Q}$. This procedure, as we may see, involves the intervention of addition and subtraction operations, associated with the theory of proportions, as presented above. It is observed that it is very important to work with the ordered set (x, y) and not with the set of numbers $\{x, y\}$, because it matters the order in which the elements *x* and *y* are arranged, because to the couple (y, x) will correspond by association to the number $y - x \in \mathbb{Q}$, which differs from the general one $x–y$. These elements presuppose the idea of order or transcendental content at the level of sensibility, which we talked about above – here we are talking, for example, about non-commutativity and/or (i)rreversibility.

The pairs of elements $(x, y)$of a set *G* are obtained as elements of the Cartesian product $G \times G$ – here we have the obvious example of the unity between algebra and geometry or analytical geometry formulated by Descartes. So, to associate to each pair (x, y) of $G \times G$ an element of *G*, we have to define a function from $G \times G$ to the set *G* and thus *the notion of internal composition* law or *algebraic operation* on the *G* set appears. This function can be any function defined on $G \times G$ with values in G like this:

$$*: G \times G \to G$$
$$(x, y) \to x * y$$

is a function that associates each pair $(x, y) \in G \times G$ with an unique element $x * y \in G$. The algebraic operation can be noted with the help of other symbols, which we can set as we wish, as they are $+, -, \odot, \oplus, \otimes$, etc.

A problem that arises now is whether by composing two elements $(x, y) \in G \times G$ the element that is obtained $x * y$ is also from *G*. It thus appears necessary to introduce the notion of *stable part* or *good definition* for a law of composition. If we consider a nonempty *G* set and an operation defined on it denoted as "$*$", we will say that *G* is *stable with respect to the law* "$*$" or that it is *well defined* iff:

[29]See note 24.

$$\forall x, y \in G \Rightarrow x * y \in G.$$

Since in mathematics the most common notations are additive (addition) and multiplicative (multiplication), it would be good to understand the adaptation of terminology beyond these operations. Starting from the good definition of the operations we can endow them with properties as follows. We will consider G a nonempty set and define the operation "$*$" on G that has the properties:

a) The operation $*$ is associative if $(x * y) * z = x * (y * z), \forall x, y, z \in G.$

b) The operation $*$ is commutative if $x * y = y * x, \forall x, y \in G.$

c) The operation $*$ admits a neutral element if $\exists e \in G$ thus $x * e = e * x = x, \forall x \in G.$

d) If the operation $*$ admits the neutral element $e \in G$ an element $x \in G$ is said to be symmetrizable with respect to the $*$ law if $\exists x' \in G$ so that $x * x' = x' * x = e.$

This generalization of the elementary operations corresponds, in the Kantian scheme that I proposed above, to the (step) of the expansion of the figurative synthesis over the domain of mathematics (algebra and geometry) through the construction in pure intuition, because both the Cartesian product and the *law of internal composition* determined as a *stable part* are more than the simple assembly of whole units.

Having the above properties, we will be able to move on to the notion of *algebraic structure*. Algebraic structure in mathematics in general means a non-empty set endowed with one or more operations that satisfy certain axioms, among which the basic ones are the ones stated above. They are mainly derivable from the Eudoxian theory of proportions from Sutherland's interpretation, synthesized by us in the first part, properties that allow such developments.

We remind here, briefly, that the definition of an algebraic structure is an axiomatic system – that, as we saw, in Kant, requires the necessity of pure intuition, which even Sutherland justified in this way.

We will focus on the algebraic notion we are concerned with at the end of this article, namely the *group*, in order to finally cover the notions of morphism and isomorphism of groups.

If we consider the nonempty set G and an operation "$*$" defined on G, a couple $(G,*)$ is called a *group* that satisfies the properties of: associativity, neutral element and admits symmetrizable elements, and in addition if it admits the property of commutativity, the couple $(G,*)$ is called a *commutative* (*abelian*) *group*.

We note that, in our scheme, the "group" implies obtaining a more complex algebraic structure by combining, developing and expanding the above elements. As a structure founded in the mentioned elements, these "extensions" presuppose the development of the group based on the generalized composition operations and developed from the elementary operations; here we have, in correspondence, the "steps" of figurative synthesis. As a mere radiography of this structure, the group assumes compositional operations based on figurative synthesis, the strict homogeneity of numerical differentiation defined by Sutherland, etc.

Starting from the algebraic structure of group we will show in our scheme how different groups are related to each other. In mathematics, this correlation is necessary due to the difficulty of studying the properties of some of them, a difficulty arising from the way of defining the operation "$*$" and/or the set G on which the operation is defined. In what follows, we will introduce the notion of *morphism of group*, which would correspond to Kant's task to expand mathematical knowledge through *a priori synthesis*.

Let the groups $(G,*)$ and $(G',\circ)$and the application $f: G \to G'$ with the property that:

$$f(x * y) = f(x) \circ f(y), \forall x, y \in G,$$

is called a *morphism of groups*. If in addition f, it is a bijective function then f it is an *isomorphism of groups*. If the sets G and G', on which the two groups are defined, are finite sets then the groups $(G,*)$ and $(G',\circ)$are finite groups. If they have the same number of elements (same order) then the two groups are isomorphic if they are equally "organized". We must understand that two isomorphic groups are "identical" in respect with the algebraic properties they possess. For example, if one is commutative then the other is also commutative, etc. Direct and inverse isomorphism (i.e. f^{-1}) "transport" one algebraic structure into the other. This means that all properties from one group are transferred to the other group.

In mathematics, the recognition of a group is done by highlighting a group isomorphic to it, which is a well-studied group whose properties are well known. One can thus reach a *degree of abstraction* as follows: *all groups isomorphic to each other behave the same*. So they constitute a "type" of groups, meaning that they can assimilate with only one of them, detaching from the G set and the operation "$*$" of each individual group, so its concrete *identity* no longer matters. In conclusion, the determination of a group is done up to an isomorphism, or in the literature it is also said, *abstracting from an isomorphism*. In practice, we try to put in an isomorphic relationship some groups that are difficult to study with groups for which we have all the already known properties.

Therefore, we understand the "transportation" of one algebraic structure into another not only as an expression, in Kant, of the *a priori* synthetic extension of mathematics, but also as a more complex form of the transfer of *order* and *transcendental content* (of sui generis lawfulness) from the "route" of the original synthetic unity of transcendental apperception, mentioned above. The presence of the homogeneous multiple is also identified, defined as "strict homogeneity" by Sutherland, where the identity of each individual group is lost through isomorphism, reaching a higher degree of abstraction. The determination of a group up to isomorphism is common to both algebra and geometry, such results and the more complex mathematical tools used can be justified in Kant's philosophy of mathematics, as we have tried to show, by reconsidering Sutherland's interpretation in the wider Kantian framework that we have already described.