SPACE AND ITS DEFORMATIONS IN THE WORKS OF NICOLE ORESME AND BERNHARD RIEMANN

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Abstract. Quite unexpectedly, the definition of curvature appears for the first time at the middle of the 14th century, in the works of Nicole Oresme. Oresme's reasons to introduce his concept of curvitas are not related to the mathematical motivations that later authors pursued to investigate deformations of shape, and together with it the modern concept of curvature. The assertion we propose is that Oresme introduced the idea of investigating shape (called in his work "configurations") at the same time with introducing the idea of curvature as deformation of shape, which makes the so-called (by its author) doctrine of configurations a striking contribution to the medieval thinking, similar in this aspect with Riemann's fundamental contribution from 1854.

Keywords: space; curvature; manifold; continuity; Nicole Oresme.

INTRODUCTION

As far as we have been able to track back this fundamental mathematical concept, the definition of curvature appears for the first time at the middle of the 14th century, in the works of Nicole Oresme[1]. If there is any other source where the concept of curvature is discussed prior to 1351, it did not reach us and it is not known to us. What is quite impressive[2] is that Oresme's reasons to introduce the very idea of curvature is not related to the reasons later authors, including Huygens and Newton, had to introduce and investigate deformations of shape. Although Oresme's "doctrine" lacks the quantifications and characterizations that the later

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[1] Nicole Oresme, *De configurationibus qualitatum et motuum*, in *Nicole Oresme and the Medieval Geometry of Qualities and Motions, a treatise on the uniformity and difformity of intensities known as Tractatus de configurationibus qualitatum and motuum*, edited with an introduction, English translation and commentary by Marshall Clagett, The University of Wisconsin Press, 1968.

[2] Isabel M. Serrano, Bogdan D. Suceavă, "A medieval mystery: Nicole Oresme's concept of curvitas", in *Notices Am. Math. Soc.*, 62, 2015, pp. 1030–1034.

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authors employed, powered by the idea of rate of change and the apparition of differentiable calculus, the remarkable characteristic we hereby propose is that Oresme introduced the idea of investigating shape at the same time with introducing the idea of curvature as deformation of shape. This structure of Oresme's theory admits analogies with Bernhard Riemann's *Über die Hypothesen, welche der Geometrie zu Grunde liegen*, written five centuries after Oresme's investigation, and triggered by Gauss' towering influence[3]. While we propose a parallel analysis of Oresme and Riemann's contributions, we are aware that each of their separate investigations have been pursued in different times, in different intellectual contexts and were subject to very different paradigms. Thus, we focus our discussion on looking at this important duality: (A) space and (B) its deformations. In mathematical terms, it would be the same as saying that we look at a Riemannian smooth manifold, endowed with a metric, which yields a certain curvature, to characterize its shape, or rather its deformations. And we are perfectly aware that this concept pertains to the development of contemporary differential geometry.

Since the current paradigm in Riemannian geometry is rooted in investigating the concept of a smooth Riemannian manifold endowed with a metric, as a form of representing space, perhaps we should start with an important historical point.

The assertion we'll pursue in the present paper is the following: in both Oresme's *Doctrine of configurations* and in Riemann's *Über die Hypothesen…* the discussion of the idea of *shape* is done at the same time with analysing the *deformation of space*. In both constructions, we cannot separate a mathematical representation of the idea of space from its deformation. That's why the idea of curvature is so important, from both mathematical and philosophical standpoint. When we are analysing curvature, we are actually reflecting on the very idea of space.

BERTRAND RUSSELL ON THE QUESTION OF SPACE

We argue that we cannot separate any discussion of Oresme's *doctrine of configurations* from the question of space. The origin of the theme, however, has different roots than the Aristotle's tradition, where Oresme found his inspiration. Bertrand Russell reflects on the following important fragment, from Plato's *Timaeus*:

> There is one kind of being which is always the same, uncreated and indestructible, never receiving anything into itself from without, not itself going out to any other, but invisible and imperceptible by any sense, and of which the contemplation is granted to intelligence only. And there is another nature of the same name with it, and like to it, perceived by sense, created, always in motion, becoming in place and again vanishing out of place, which

[3] Bernhard Riemann, "On the Hypotheses which lie at the Bases of Geometry", translated by Willian Kingdon Clifford, in *Nature*, 8, 1873, pp. 14–17, 36–37.

> is apprehended by opinion and sense. And there is a third nature, which is space, and is eternal, and admits not of destruction and provides a home for all created things, and is apprehended without the help of sense, by a kind of spurious reason, and is hardly real; which we beholding as in a dream, say of all existence that it must of necessity be in some place and occupy a space, but that what is neither in heaven nor in earth has no existence.[4]

About this passage, Bertrand Russell writes:

> This is a very difficult passage, which I do not pretend to understand at all fully. The theory expressed must, I think, have arisen from reflection on geometry, which appeared to be a matter of pure reason, like arithmetic, and yet had to do with space, which was an aspect of the sensible world. In general it is fanciful to find analogies with later philosophers, but I cannot help thinking that Kant must have liked this view of space, as one having an affinity with his own.[5]

The question we discuss therefore could be phrased as: "what is space?" We argue that any discussion on the history of this theme should include Nicole Oresme's theory, even if we are hereby proposing an author who did not use a precise computational tool to determine curvature for planar curves, space curves, or for surfaces, as later authors did, either by using the tools of calculus, or later on vectorial or tensorial calculus. Nevertheless, for this particular topic, Oresme's contribution is way ahead of his time, as we hope we'll be able to prove below.

SPACE AND ITS DEFORMATIONS IN THE WORK OF RIEMANN

The peak achievement that produced a satisfactory clarification of representing space, together with its deformations, is found in the work of Bernhard Riemann, at the middle of the 19th century[6], which means precisely five centuries after Oresme. It was Riemann who proposed the idea of manifold (translated by Clifford as *manifoldness*), as a mathematical representation of the idea of space.

For the reader familiar with the contemporary Riemannian geometry, returning to the source and reading Riemann always represents a very profound experience: all the potential of the theory is there, in the original work, *in nuce*. The idea of differentiable atlas is there, the description of curvature is also there. We proposed above to examine the existence of this pair made up of (A) the representation of space; and (B) the idea of space's deformation. This pairing is clearly described by Riemann in the following terms: Having constructed the notion of a manifoldness

[4] Bertrand Russell, *The History of Western Philosophy*, A Touchstone Book, Simon and Schuster, 1972, original printing 1945; see p. 146.

[5] *Ibidem*.

[6] B. Riemann, *idem.*

of *n* dimensions, and found that its true character consists in the property that the determination of position in it may be reduced to *n* determinations of magnitude, we come to the second of the problems proposed above, viz. the study of the measure-relations of which such a manifoldness is capable, and of the conditions which suffice to determine them. These measure-relations can only be studied in abstract notions of quantity, and their dependence on one another can only be represented by formulae. On certain assumptions, however, they are decomposable into relations which, taken separately, are capable of geometric representation; and thus it becomes possible to express geometrically the calculated results. In this way, to come to solid ground, we cannot, it is true, avoid abstract considerations in our formulae, but at least the results of calculation may subsequently be presented in a geometric form. The foundations of these two parts of the question are established in the celebrated memoir of Gauss, *Disquisitiones generales circa superficies curvas.*

One can only admire Riemann's precision in describing here the full potential of Riemannian geometry, even though historically we are here before the birth of vector calculus, before tensor calculus, before the control that the so-called Riemann-Christoffel tensor provides over the concept of curvature. We argue that in Riemann's construction the duality space & deformation is there, and that is quintessential for the developing of the idea. It amounts to saying that we cannot discuss the idea of space without fully grasping what its deformation is, and that this deformation is quantified by the idea of curvature. That quantification is indeed rooted in Gauss' work from 1827, exactly as Riemann pointed out[7].

ORESME AND HIS DOCTRINE OF CONFIGURATIONS

After we have seen the idea fully described in Riemann's work, we will step back in time to examine with utmost attention Nicole Oresme's perspective. It can be asserted that many of Oresme's philosophical ideas are due to the increasing impact that the works of Aristotle had in many scholarly circles in the Middle Ages, especially in the period that preceded the Black Death[8]. Before the Recovery of Aristotle[9], there were only two of Aristotle's books translated into Latin: *Categories* and *On Interpretation*. St. Augustine (354–430) describes in his *Confessions* (IV.xvi.28) how he was overwhelmed by a reading of the *Categories*

[7] A thorough discussion of Gauss' *Disquisitiones* is in Michael Spivak, *A Comprehensive Introduction to Differential Geometry*, vol. 2, 3rd edition, Houston, Publish or Perish, 1999.

[8] See Edward Grant, *The Foundations of Modern Science in the Middle Ages: Their Religious, Institutional and Intellectual Contexts*, Cambridge Studies in the History of Science, Cambridge University Press, 1996. And also by Edward Grant, *Science and Religion, 400 B.C. to A.D. 1550: From Aristotle to Copernicus*, Johns Hopkins University Press, 2006.

[9] R.E. Rubinstein, *Aristotle's Children: How Christians, Muslims, and Jews Rediscovered Ancient Wisdom and Illuminated the Middle Ages*, Harvest Books, 2004.

at the age of 20. For the theme we discuss, one important detail is hidden in Aristotle's text: the idea of continuous measure. It can be debated whether Aristotle understood by "measure" the same thing we do today, but nevertheless in Chapter 6 of the *Categories* we read (and we underscore in italics the idea of continuity): Of Quantity, one kind is discrete, and another *continuous*; the one consists of parts, holding position with respect to each other, but the other of parts, which have not that position. Discrete quantity is, as number and sentence, but *continuous*, as line, superficies, body, besides place and time.

It is quite important that "space" and "time" are listed here, because this is the foundational stone on which Oresme builds. This discussion on continuity of space seems to have been the inspiration for Oresme in his construction of his *doctrine of configurations,* as we will further analyse below.

CONTINUITY IN SEVERAL DIMENSIONS: WHAT GAUSS WROTE ABOUT IT

If we are reading Oresme, right at the beginning of his *De configurationibus* we find the following comment: "Every measurable thing except numbers is imagined in the manner of continuous quantity." We are centuries before the important distinction between measurability and continuity, which after the modernist transformation of mathematics (as the period 1890–1930 is called) represent two very different matters. However, this is an important nuance: Oresme thinks in the terms of the Western European paradigm of Aristotle's Recovery, he bases his assessments on the Magister's thoughts and once he found in Aristotle reflections on continuity, he builds on them. On the other hand, to better get a feeling on the concept, if we pursue the idea of continuity in Gauss' work, we find in *Disquisitiones*, part 3, the following description:

> A curved surface is said to possess continuous curvature at one of its points *A,* if the directions of all the straight lines drawn from *A* to points of the surface at an infinitely small distance from *A* are deflected infinitely little from one and the same plane passing through *A*. This plane is said to *touch* the surface at the point *A.* If this condition is not satisfied for any point, the continuity of the curvature is here interrupted, as happens, for example, at the vertex of a cone.[10]

About this passage, Michael Spivak proposes a rather critical comment: "This section merely defines (or tries to define) a differentiable surface, and its tangent plane at a point." Gauss' formalism might not live up to the contemporary standards of notations, but the idea is fully there, and the very fact that Gauss is

[10] See Michael Spivak, *idem*, p. 63.

discussing the class of curvature is truly remarkable (more precisely: the failure of curvature of being continuous at a point, as an expression of class). Discussing the continuity of a characterization of shape is an important point, and Gauss does not miss it. There are curvature invariants that do not behave smoothly, and we should not take the continuity of the characterization for granted, as it is the case with the so-called Chen's δ-curvature invariants, which depend on the infimum and supremum of the values taken by the Riemann-Christoffel tensor in a given planar direction[11].

L. EULER DID NOT WORK WITH THE CURVATURE OF SURFACES

One generation before Gauss, the question of curvature of a surface at a given point was far from being settled. To better understand how the concept of curvature of surfaces was viewed before Gauss, we recall that Leonhard Euler wrote in 1763 that we cannot define the curvature for surfaces. He writes:

> Pour connaître la courbure des lignes courbes, la détermination du rayon osculateur en fournit la plus juste mesure, en nous présentant pour chaque point de la courbe un cercle, dont la courbure est précisément la même. Mais, quand on demande la courbure d'une surface, la question est fort équivoque, et point du tout susceptible d'une réponse absolue, comme dans le cas précédent.[12]

Today we fully understand the geometric detail that eluded Euler. It was the situation when the tangent plane at a point to a surface is intersecting the surface, that is the case when the surface is hyperbolic at that point, as it happens for example at every point of the catenoid. What Euler hoped to see materialized was the idea of approximation of the surface with a sphere, an extension of Newton's idea of osculating circle approximating locally the curve. However, it was not this idea that produced the definition of the curvature at a point of a surface, as we can see by reading Gauss' *Disquisitiones.* but a combination of ideas including small infinitesimals and limiting processes, all of which were at Euler's disposal. It was Gauss' vision that generated the curvature of surfaces, and it was Riemann's effort who extended this concept to *n*-dimensional representation of space.

[11] Introduced in Bang-Yen Chen, "Some pinching and classification theorems for minimal submanifolds", *Arch. Math.* 60, No. 6, 1993, pp. 568–578. See also, for a more recent overview on the development of the theory, Bang-Yen Chen, *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, World Scientific, 2011.

[12] Leonhard Euler, *Recherches sur la courbure des surfaces* [*Research into the curvature of surfaces*], in *Mémoires de l'Académie des Sciences de Berlin* 16, 1767, pp. 119–143.

Quite surprisingly, Oresme did not possess any small infinities, and that did not stop him to bring forth into his analysis of his *configurations* the idea of *curvitas.* The *doctrine of configurations* is lacking quantification, but is not lacking vision and geometric intuition, irrespective of the author's singularity in his time.

READING NICOLE ORESME

Nicole Orseme was born around 1320 in the village of Allemagne, near Caen, today Fleury-sur-Orne. He later was a "bursar" of the College of Navarre from 1348 to October 4, 1356, when he became a Master. The College of Navarre, established by Queen Joan I of Navarre in 1305, focused on teaching the Arts, Philosophy, and Theology to intellectuals who could not afford to attend the University of Paris. Oresme's major was in Theology, and not Mathematics. Oresme studied, among others, with Jean Buridan and Albert of Saxony, and he definitely had good scholarly models around him. It was at this institution where he wrote his most important works, *De proportionibus proportionum*, which is of particular importance for the history of mathematics, or *Ad pauca respicientes* of interest for the history of ideas in celestial mechanics.

Orseme remained Master of the College until December 4, 1361, when he was forced to resign. On November 23, 1362 he became a canon of the Rouen Cathedral and on March 18, 1364 he was promoted to Dean of the Cathedral. Sometime before 1370 he became one of Charles V's (1364–1380) chaplains and at the king's request he translated from Latin into French Aristotle's *Ethics* (1370) and *Politics* as well as *Economics*. For any biographical details on N. Oresme's life and for other interpretations on his work we refer to either G.W. Copeland's work[13] or to other sources[14].

To fully describe his theory, Orseme begins his *De configurationibus* with the following clarification: "Every measurable thing except numbers is imagined in the manner of continuous quantity." Then he pursues a discussion of the latitude and longitude of qualities, followed with the presentation of their quantity. He leads into his argument that qualities can be "figured". He spends several chapters discussing suitability of figures and shape of various particular cases. One important distinction appears in chapter I.xi, where Oresme examines[15] the differences between uniform and difform qualities. He continues his focus on this topic in I.xiv with a discussion of "simple difform difformity", which is of two kinds: simple and

[13] G.W. Coopland, *Nicole Oresme and the Astrologers. A Study of his Livre de Divinacions*, Cambridge, Mass., Harvard University Press, 1952.

[14] See Dirk J. Struik, *A Source Book in Mathematics: 1200-1800*, Cambridge, Mass., Harvard University Press, 1969. For a recent analysis, see Bogdan D. Suceavă, Anael Verdugo, "Nicole Oresme's Quest towards the Realm of Reality: Are There Any Themes of Mathematical Modeling Present in His Works?", *Creat. Math. Inform.*, 32, No. 2, 2023, pp. 237–246.

[15] We discussed these details also in Isabel M. Serrano, Bogdan D. Suceavă, *idem.*

composite. The matters could have been discussed differently, if Oresme had better algebraic and geometric tools. In I.xv he begins describing four kinds of simple difform difformity, which are explained by drawing graphs. This work represents a very interesting early discussion of convexity and concavity. After this extensive discussion, performed without any algebraic notations, Oresme approaches "surface quality", which represents actually his way of working with the concept of space. In the *doctrine of configurations* we do have representations of space. In chapers I.xix, I.xx, and I.xxi, Oresme introduces the concept of curvature.

A particular case in his so-called *doctrine of configurations* represents curvature (chapter I.xx), endowed with "both extension and intensity". When Oresme does that, he relates the concept of curvature to the category of space. Oresme writes (in M. Clagett's translation): "we do not know with what, or with regard to what, the intensity of curvature is measured. For now it appears to me that there are only two [possible] ways [to speak of the measure of curvature]. The first is that the increase in curvature is a function of its departure from straightness, i.e. of its distance from straightness. This is [to be measured] by the quantity of the angle constituted of a straight line and a curve, e.g. an angle of contingence or perhaps another angle also constructed from a straight line and a curve."

We pointed out in an earlier work[16] that this very intuitive description is very consistent with the modern study of signed curvature and its relationship with the change of directional angle with respect to arc length. This intuitive idea is fully captured in this paragraph. But Nicole Oresme creates a more precise explanation. He continues to write specifically that the curvature of the circle is the inverse of its radius in chapter I.xxi, where Oresme cites Aristotle's *On Curved Surfaces*, which therefore must have been known to him. He delves in more into this concept by covering more intricate examples of curves: "difform curvature is composed of an infinite number of parts of different nature and unrelatable [to each other]" (I.xx).

Quite remarkably, Oresme has also counterexamples of "qualities" that do not fit his theory. He writes that "a quality of an indivisible subject, such as a soul or an angel, does not have extension". In fact, we are saying today that there are processes that do not admit a mathematical model, and we accept this fact.

Additionally, Oresme's approach on velocity is quite modern, in chapter II.iv., where he discusses "diverse ways of [considering] velocity". He is comparing the displacement of material points with the slow growth of trees. Nevertheless, the concept of space pertains implicitly to this discussion. Oresme picks up the conversation on velocity later on, in Part III of his treatise, where III.v is dedicated to an analysis "On the measure of uniform qualities and velocities".

Continuity is essential in a discussion as in Oresme's II.v, titled "On certain other successions in motion". He points out that "every velocity is capable of being increased in intensity and decreased in intensity." He uses "continua" in a sentence

[16] Isabel M. Serrano, Bogdan D. Suceavă, *idem.*

like “its continuous increase in intensity is called acceleration”. All these definitions represent particular cases of “configurations”, as this “doctrine” seems to be truly a precursor of the theory of functions. However, no vertical line test to establish when a relation is a function, and, unfortunately, no quantifications; in his *doctrine* we are centuries ahead of that. In chapter II.ix., Oresme proposes the comparison of two difformities. The “measure and intention to infinity of certain difformities” appears in III.viii. The discussion of finite vs. infinite appears in III.xi. and III.xii and it exceeds anything Oresme could have had inspiration about from Aristotle. In any case, all of these ideas showcase that Oresme is discussing representations of space, and his graph-like “configurations” are in fact ways of discussing space.

CONCLUSIONS

The fact that Nicole Oresme produced a discussion of curvature as a measure of deformation of shape is quite remarkable, and this intellectual achievement stands out in the constellation of medieval thought. We compared Oresme’s approach with the works of later authors, and we discovered how advanced his thinking was, not just for the 14th century, but for later centuries as well. If we are to discuss the idea of deformation of space, perhaps the most appropriate comparison of Oresme’s *doctrine* should be with Riemann’s construction, which is the starting point in our the contemporary paradigm in discussing a mathematical representation of space. The fact that Oresme’s substantive work was written at College de Navarre during the Black Death, all during the Hundred Year War, have been all meaningful historical factors that delayed the dissemination of his thinking, and reduced any hopes of seeing his work continued by other scholars in direct academic filiation.