

# FORM-INDEPENDENCE IN THE SORITES PARADOX

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**Abstract.** While the Sorites paradox is often presented in a propositional conditional form, it can be recast using disjunctive, conjunctive, or modus tollens syllogism. This paper challenges Kit Fine’s claim that only the conjunctive form is motivated. All four propositional forms are effective because they express three deeper, first-order principles: (1) the familiar principle of Tolerance, (2) the often-implicit assumption of Monotonicity, and (3) a novel claim I call Local Interdefinability (LI). LI is the thesis – motivated by contextualist empirical observations and by the salience of vague contraries – that for adjacent members of a soritical series, vague contraries (such as “tall” and “short”) function as contradictories. LI lets the conjunctive form proceed without any explicit elimination of double negation. Moreover, Kit Fine’s case against the conditional form loses its force since LI covers the gap between the conjunctive and conditional forms, and his case against the disjunctive form is undermined by the rejection of Monotonicity. The paper concludes that the unity of the Sorites paradox lies above the propositional level.

**Keywords:** sorites, vagueness, propositional logic, compatibilism, tolerance.

## 1. INTRODUCTION

The Ancient Greek paradox known as the Heap starts by stating that one grain of corn does not make a heap, and that no difference of one grain can transform a non-heap into a heap. Therefore, no number of grains can make a heap<sup>1</sup>. Let us call such discursive pre-logical forms the “*intuitive*” form of the Sorites paradox.

Nowadays, the Sorites paradox is commonly<sup>2</sup> presented with a propositional and an inductive form.

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<sup>1</sup> Also, this one from Diogenes Laertius: “It is not the case that two are few and three are not also. It is not the case that these are and four are not also (and so on up to ten thousand). But two are few: therefore, ten thousand are also” Diogenes Laertius, “Lives of the philosophers” in Rosanna Keefe, Peter Smith (eds.), *Vagueness: A reader*, MIT Press, 1996, p. 58.

<sup>2</sup> Rosanna Keefe, Peter Smith, “Introduction: theories of vagueness,” in *Vagueness: A reader*, MIT Press, 1996, pp. 10–11.

## (CS) THE CONDITIONAL FORM

(CS.m) A man of 200 cm is tall.	A
(CS.M.1) If a man of 200 cm is tall, so is one of 199 cm.	$A \supset M_1$
(CS.M.2) If a man of 199 cm is tall, so is one of 198 cm.	$M_1 \supset M_2$
...	...
(CS.M.100) If a man of 101 cm is tall, so is one of 100 cm.	$M_{99} \supset B$
(CS.C) A man of 100 cm is tall.	B

## (IS) THE INDUCTIVE FORM

(IS.m) A man of 200 cm is tall.	$Ta_{200}$
(IS.M) For any number of cm, if a man of that number is tall, so is a man of that number minus one.	$\forall n, n \in \mathbb{N}. Ta_n \supset Ta_{n-1}$
(IS.C) A man of 100 cm is tall.	$Ta_{100}$

There is a *minor premise*, i.e. “.m”, which states a certainty: that 200 cm make a tall man. Then a single *major premise* indicated by “.M” (in the case of the inductive form) or a finite number of steps playing its role. The paradoxical conclusion is indicated by “.C”. The propositional form above can be further classified according to the rule of inference it employs, such as modus ponens – (CS) above –, conjunctive, disjunctive and a negative form using modus tollens.

The conditional form is by far the most discussed, so that major non-classical treatments of the conditional have been applied to vagueness and claimed to solve the paradox by this very fact. Fuzzy logicians adopted Łukasiewicz’s redefinition of the conditional, defusing the conditional Sorites as invalid yet highly believable<sup>3</sup>. Then it was remarked that the Sorites could advance by, say, conjunctive syllogism, which makes the standard fuzzy explanation highly dubious<sup>4</sup>.

In his recent introduction of compatibilism<sup>5</sup>, Kit Fine attempts a similar approach. He proposes that a single propositional reading, namely the conjunctive one, is motivated (yet invalid), while the conditional and disjunctive forms are unmotivated (yet valid).

<sup>3</sup> The fuzzy conditional takes value 1 whenever the consequent is at least as true as the antecedent and otherwise takes 1 minus the difference between the antecedent’s and consequent’s degrees of truth. If the antecedent has 0.8 and the consequent has 0.7, the conditional will have 0.9, but if the consequent has 0.82, it will have 1. So you can lose 0.01 truth from 0.99 to 0.98, 0.01 from 0.98 to 0.97 and so on, getting to 0 in small steps, each of 0.99 validity, giving an intuitive story of the believability of the paradox, which is not readily available for other modes of inference. Kenton Machina, “Truth, belief and vagueness,” in Rosanna Keefe, Peter Smith (eds.), *Vagueness: A reader*, MIT Press, 1996, p. 197.

<sup>4</sup> Nicholas J.J. Smith, *Vagueness and degrees of truth*, Oxford University Press, 2008, pp. 265–268.

<sup>5</sup> Kit Fine, “The possibility of vagueness,” *Synthese*, vol. 194, nr. 10, 2017.

This paper studies why the paradox – once expressed propositionally – is able to advance in these four (and more) modes of inference. Section 2 extracts three logico-philosophical claims embedded in the intuitive form of the paradox: the well-known *Tolerance* claim (TOL) and its converse which I call *Monotonicity* (MON), plus a *Local Interdefinability* (LI) of soritical contraries (e.g. heap – non-heap, tall – short) which elucidates at a semantic level the salience of such contraries. Of course, it is accepted that *TOL* needs weakening, as most philosophical logics of vagueness propose, since it is the main culprit leading to paradox. But *MON*, a heuristic often assumed wholesale and embedded into various non-classical semantics, has prima facie counterexamples due to multidimensionality, so it should not be treated as certain. *LI*, on the other hand, is the weakest claim with empirical support and no counterexamples, so we accept it in order to account for the salience of vague contraries. In Section 3, Fine’s arguments against the motivation of the conditional and disjunctive forms of the Sorites paradox are rejected, with the conclusion that all propositional modes of inference deliver a paradox. Section 4 argues that any theory of the Sorites should tackle directly such above-zero-order claims, as it is there that vagueness lives.

## 2. THE INTUITIVE FORM AND THREE FIRST-ORDER CLAIMS

Let us start with the intuitive discursive form of the Heap above. We have:

- (i) A soritical series: closely placed increasing (or decreasing) quantities at each step (i.e. one grain, two grains, ...).
- (ii) A claim which extends the main predicate along the soritical series in both directions (i.e. “no difference of one grain can transform a non-heap into a heap”).
- (iii) For each step of the series, an affirmation of the main predicate or of the opposite predicate (“one grain of corn does not make a heap”).
- (iv) A doublet: the main predicate (i.e. “heap”) and an opposite predicate (i.e. “non-heap”).

We will now propose two first-order claims unifying (i), (ii) and (iii), and a third one justifying (iv).

### A) TOLERANCE AND MONOTONICITY

The role of the major premise (.M) of the propositional and inductive forms of the Sorites was long thought to be the embedding of a so-called “principle of tolerance”. This is our (ii) above applied to the series of (i), so that it generates the affirmations of (iii). It is the notion that a small difference is not compatible with the switch from a negative to a positive case or the converse. Crispin Wright wrote:

... we encounter the feature of a certain *tolerance* in the concepts respectively involved, a notion of a degree of change too small to make any difference, as it were. There are degrees of change in point of size, maturity and colour which are insufficient to alter the justice with which some specific predicate of size, maturity or colour is applied.<sup>6</sup>

Thus, the claim extending the main predicate along the soritical series can be called *Tolerance (TOL)*, but there is a complication. The series can be ascending (from smaller to larger quantities) or descending (from larger to smaller), while *TOL* is intended to go in one direction only, in which the predicate used in the minor premise of the paradox has weakening application. For “bald”, *TOL* will be increasing (from one hair to two hairs, to three hairs, etc.) while for “tall” it would be decreasing (from 200 cm to 199 cm, to 198 cm, etc.).

So, there is a paired claim alongside *TOL*, which I call *Monotonicity (MON)*, which goes the other way, the strengthening direction of the predicate. This claim is an extremely common assumption of philosophers of vagueness, traceable to the *internal penumbral connections* in Fine’s 1975 first approach to the topic of vagueness<sup>7</sup>. It says that the distribution of the property among the objects is monotone with the series ordering. As height increases, there can presumably be no not-tall person at the same or previous rank with a tall person. But no formal rule requires it, any more than it requires *TOL*. “Tall” is a monadic predicate and “having more cm”<sup>8</sup> is a relation i.e. binary predicate, they are assigned independently.

It seems to be a useful heuristic, and it was embedded in many semantics of vagueness<sup>9</sup>. But its plausibility depends on idealization, comparison classes and unidimensionality. Indeed, as things stand in the world, there are now many short persons (say some men in New York) of more cm than some tall persons (say some women of Seoul), i.e. without specifying a class of comparison. And someone with an impressively voluminous hairstyle may be tall even if they have lower scalp-to-toe measurements than a stooped short person<sup>10</sup>. Such multidimensionality – and

<sup>6</sup> Crispin Wright, “On the Coherence of Vague Predicates”, *Synthese*, vol. 30, nr. 3/4, 1975, p. 333.

<sup>7</sup> “If Herbert is to be bald, then so is the man with fewer hairs on his head.” Kit Fine, “Vagueness, truth and logic,” *Synthese*, vol. 30, nr. 3/4, 1975, p. 276.

<sup>8</sup> The same argument applies to “tall” and “taller”, the grammatical appearance is not a logical rule.

<sup>9</sup> For example any affirmation of this principle is given maximum *verity* in Dorothy Edgington, “Vagueness by degrees,” in Rosanna Keefe, Peter Smith (eds.), *Vagueness: A reader*, MIT Press, 1996.

<sup>10</sup> Previous mentions of this possibility in the literature, without taking it seriously, include “Now ‘taller’ does indeed seem more precise than ‘tall’. But it does not seem perfectly precise; stoops and curly scalps may produce borderline cases even for it.” Timothy Williamson, *Vagueness*, London, Routledge, 1994, p. 156. For an extended discussion of the effect of monotonicity on logics for vagueness see Marian Călborean, “Paths Through the Fog: A Comparative Analysis of Ten Philosophical Logics for Vagueness,” *Revue roumaine de philosophie*, vol. 69, nr. 1, 2025.

maybe other reasons independent of comparison classes<sup>11</sup> – defeats MON. Since it is not a logical law, from a logical-agnostic point of view, it instantiates the same claim as Wright’s Tolerance, namely “*No difference one cm apart*”, either ascending or descending. I will argue that Fine should not rely on a defeasible heuristic in claiming that a mode of inference (i.e. disjunctive syllogism) is unmotivated.

Regarding the series of affirmations (iii) above, the inductive form, in contrast to propositional forms of the Sorites, does away with the precise number of steps needed to get between the endpoints. It assumes the standard definition of natural numbers and states the inductive property for any  $n \in \mathbb{N}$ . This means that ‘ $a_n$ ’ in the ‘ $Ta_n$ ’ of (IS.M) is referring to a height, not to persons<sup>12</sup>. It is assumed that there is such a height for each natural number such that the successor relationship holds. Then, the soritical property is hereditary in the successor relation<sup>13</sup>. This means that if a height of a number has it, so does the height of the subsequent number. Yet it is illuminating to reflect on an inductive form for a non-integer soritical step, such as  $\pi/4$  cm<sup>14</sup>. Since induction is made on  $\mathbb{N}$ , the definition of ‘ $a_n$ ’ would need to be modified to include multiplying  $\pi/4$  by  $n$ . The starting and ending point would change too. If started at  $64\pi$  (201.0619...) cm, the end might be, for example,  $32\pi$  (100.5309...) cm. If started at 200 cm as before, the end might be  $200 - 32\pi$  (99.4690...) cm. We see that the size of the soritical step determines the number of steps and the end point, given the starting point. But not even the inductive form captures logically this relation.

Therefore, we will settle on retaining the first-order form of the inductive form while giving up the distracting mathematical assumptions. The claim of TOL for tall is that two people, not heights, one descending cm apart from each other,

<sup>11</sup> One may argue that you can hold all the comparison classes constant and still have a soritical series *ceteris paribus* with cases differing in one dimension only, for example a soritical series of reddish patches. But that assumes that real individuals with their myriad properties may differ in one attribute only, which is a) not realistic for physical objects (different colors are usually caused by different physical makeup) and b) a simplifying semantic assumption (maybe the definition of ‘red’ includes a long disjunction of precisely the conjunctions of all properties red objects have individually). For elaboration see Marian Călborean, “The nature and logic of vagueness,” (PhD Thesis, University of Bucharest, 2020), pp. 40–43.

<sup>12</sup> You cannot say that  $n$  ranges over  $\mathbb{N}$  and then say that  $a_n$  in (IS) above refers to a person without specifying how each person is in  $\mathbb{N}$ . To wit, similar to Frege, Hyde speaks of the relation as the “addition of one hair” to a person, but this would not guarantee mathematical induction. Dominic Hyde, “The Sorites Paradox,” in Giuseppina Ronzitti (ed.), *Vagueness: A Guide*, Springer Verlag, 2011, p. 13.

<sup>13</sup> For the first definition of hereditary property, see #24 formula 69 in Gottlob Frege, “Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens,” in Jean van Heijenoort (ed.), *From Frege to Gödel: a source book in mathematical logic, 1879–1931*, Harvard University Press, 2002, p. 55. He uses it in #27, formula 81, at page 62. Incidentally, Frege’s discussion of formula 81 is one of the first modern discussions of the Sorites paradox.

<sup>14</sup> While measurements are rational numbers and the rational/irrational distinction does not affect my argument, you can give a sense to walking  $\pi$  meters by saying you walked half the circumference of a circle drawn on asphalt with a radius of one meter. Walking  $10\pi$  meters would be walking that circumference five times and so on.

cannot be one a negative and the other a positive case of the predicate. We keep the hereditary nature of “*tall*” – T but note by “S” being one cm more, not the successor relation. With “S” the **relation** of decreasing (by cm) adjacency:

$$(TOL) \forall xy . Sxy \supset \neg(Tx \wedge \neg Ty)^{15}$$

This says that if there are two men, the first of a little more cm than the second, it cannot be that the first is tall and the second is not.

Of course, the exact logical form is shared with the claim of *Monotonicity* discussed above. With “S” the same relation of decreasing (by cm) adjacency:

$$(MON) \forall xy . Syx \supset \neg(Tx \wedge \neg Ty)$$

Thus, the difference between TOL and MON is that while the former is *weakening* the predicate application i.e. the relation is from more cm to less cm for “tall” or from fewer hairs to more hairs for “bald”, MON is *strengthening*, i.e. from less or equal number of cm to equal or more cm for “tall” and from more or equal number of hairs to equal or less hairs for “bald”. Both in a likelihood sense, since none of them are logical laws. The former creates the Sorites paradox, while the latter is usually taken as obvious, despite prima facie counterexamples e.g. if you have premature aging syndrome, you will be old at an age where most others are young. We can express them using a **discrete** total preorder, such as  $\leq$  (read “to be of lower or equal number of cm”). Let  $Adj_{\leq}(x,y)$  mean that  $x < y$  and there is no  $z$  such that  $x < z < y$ . Then:

$$(TOL_{\leq}) \forall xy . Adj_{\leq}(y,x) \supset \neg(Tx \wedge \neg Ty)$$

$$(MON_{\leq}) \forall xy . Adj_{\leq}(x,y) \supset \neg(Tx \wedge \neg Ty)$$

We split Wright’s principle of tolerance into two separate claims, the weakening TOL and the strengthening MON. They unify the first three elements of the intuitive form of the paradox: they assume a soritical binary relation, together they extend the predicate in both directions, generating the series of affirmations about each element in the series by instantiation.

## B) SORITICAL DOUBLETS AND LOCAL INTERDEFINABILITY

What remains unexplained is the salient presence of soritical doublets in the intuitive discursive forms of the Sorites. We can call such pairs “doublets”. “Tall” – “short”, “bald” – “hirsute”, “heap” – “non-heap” are such standard pairs. Mark Sainsbury writes:

<sup>15</sup> It can be checked that this is classically equivalent to Frege’s formulation:  $\forall xy . Sxy \wedge Tx \supset Ty$ .

“Boundaryless concepts tend to come in systems of contraries: opposed pairs like child/adult, hot/cold, weak/strong, true/false, and the more complex systems exemplified by our color terms”<sup>16</sup>.

Sainsbury calls them contraries, not contradictories, since some major treatments of vagueness depend on the affirmation of a gap (or glut) inside the doublet. Indeed, if we define their relation as full interdefinability:

$$(DEF) \forall x . Tall(x) \leftrightarrow \neg Short(x) \text{ and } \forall x . Short(x) \leftrightarrow \neg Tall(x),$$

any assumption of intermediate cases would collapse into a contradiction, which is unacceptable, except maybe for dialetheism. Just take  $\neg Tall(a) \wedge \neg Short(a)$  and apply DEF, you would get  $\neg Tall(a) \wedge Tall(a)$ .

On the other hand, the standard logical transcription of contrariety is an implication of the negation:

$$(CON) \forall x . Tall(x) \supset \neg Short(x) \text{ and } \forall x . Short(x) \supset \neg Tall(x).$$

This simply says that someone who is tall is not short and the converse. That is much too weak to justify the popularity of soritical doublets in the intuitive form of the paradox. There are two specific reasons.

Firstly, note that the Greek Heap above has the doublet “heap – non-heap”. And that “non-heap” is manifestly the negation of “heap”, much more directly than “short” is the negation of “tall”. So DEF may be too strong if one wants to allow something to be neither a heap nor a non-heap, but CON is trivial. An intermediate solution, even idealized, that would preserve the equivalence of DEF for many (or most) cases would do justice to the role of “non”. “Non-heap” is linked to “heap” in a stronger way than “sea” is, although certainly if something is a heap, it is not a sea.

Secondly, note that speakers have no problem in stating that if we concluded that a man of 100 cm is not tall, this means that he is short. Or if two are not non-few, then they are few. In clear cases interdefinability is appropriate. This idea was captured in the literature by treating them as fuzzy opposites, as fuzzy theory has  $|Short(a)| = 1 - |Tall(a)|$  or as polar antonyms in Christopher Kennedy’s linguistic theory of gradable adjectives<sup>17</sup>. And classic supervaluationism relates “definitely not tall” to “short” via penumbral connections, as already mentioned.

But I propose a *semantic* principle expressible with the same apparatus as TOL and MON, based on the evidence gathered by proponents of contextualism. Contextualist accounts of vagueness, such as Delia Graff’s, observe that in a forced-march Sorites, speakers do not typically jump from asserting “*Tall(a)*” to

<sup>16</sup> R.M. Sainsbury, “Concepts without boundaries,” in Rosanna Keefe, Peter Smith (eds.), *Vagueness: A reader*, MIT Press, 1996, p. 258.

<sup>17</sup> Christopher Kennedy, “Polar Opposition and the Ontology of ‘Degrees’”, *Linguistics and Philosophy*, vol. 24, nr. 1, 2001, Springer.

asserting that  $b$  is gappy<sup>18</sup>. Instead, the first move is to deny “ $Tall(a)$ ”. The principle of *Local Interdefinability (LI)* is proposed here as a semantic counterpart. As speakers move along clear cases of men being tall, they extend the complementarity of “tall” and “not short” for adjacent men. And while extending the threshold of tallness, they also extend the threshold of interdefinability between “tall” and “short”, since the conceptual firming, as discussed by Graff is the same.

The claim of *Local Interdefinability* is an idealized intermediate principle which explains the salience of soritical doublets in the intuitive form of the Sorites but also allows for (many) non-classical treatments of vagueness<sup>19</sup>. Formally, it acts as an object-language meaning postulate. With “ $\sim$ ” now a possibly dense relation of soritical adjacency:

(LI)  $\forall xy . (x \sim^{20} y \wedge Tall(x)) \supset (Tall(y) \leftrightarrow \neg Short(y) \wedge Short(y) \leftrightarrow \neg Tall(y))$  **and**  
 $\forall xy . (x \sim y \wedge Short(x)) \supset (Short(y) \leftrightarrow \neg Tall(y) \wedge Tall(y) \leftrightarrow \neg Short(y))$

What LI says is that for adjacent elements in a soritical series, if the first is tall or short, the second has “tall” and “short” as contradictories. The notation is long, but the idea is straightforward<sup>21</sup>.

That is, speakers will not grant that the first is tall without also granting that the adjacent element is not gappy. If they want to create a gap, they will create it by denying that the first is tall, which is the contextualist story above. For Graff, the corresponding argument was pragmatic, based on shifting comparison classes. If a speaker meets a man who can no longer be argued to be “tall”, the conceptual firming goes away and with it, the man will be thought of as “not tall”. That would falsify the antecedent of LI, allowing a gap. What LI forbids semantically is any setup where there is a precise point at which both “ $a$  is tall” and “ $b$  is neither tall nor short” hold, which is a weaker constraint. While TOL and MON would force  $a$  and  $b$  to be affirmed or denied together, LI only forbids gaps next to clear items. The contextualist evidence is that such speakers first deny that  $b$  – or for that matter,  $a$  – is tall. To formally authorize or block specific logical inferences, we need to promote Graff’s rule into an object-language principle. The point at which “tall” and “short” cease to be treated as contradictories is precisely the point where the Sorites reasoning turns paradoxical.

<sup>18</sup> “... let us suppose that  $a$  and  $b$  are saliently similar in respect of height, and that  $a$  is the taller of the two. Given the Similarity Constraint, they are either both in the extension of ‘tall’, both in the anti-extension of ‘tall’, or both in its extension gap.” Delia Graff, “Shifting sands: an interest-relative theory of vagueness”, *Philosophical Topics*, vol. 28, nr. 1, 2000, University of Arkansas Press, p. 62.

<sup>19</sup> For example, it is compatible with dialetheist, supervaluationist and intuitionistic semantics of vagueness.

<sup>20</sup> “ $a \sim b$ ” can be defined for dense orders, to contrast with the  $\leq$  above, by using a contextually-defined step  $\delta$ .

<sup>21</sup> If one suspects it brings back classicality, it can be replaced it with two conditionals plus a ceteris-paribus clause sensitive to comparison class, with unchanging results further down in the paper. However, I think the arguments for a semantic principle are sufficiently strong.

LI does not yield a paradox, in contrast to TOL, and does not have empirical counterexamples, in contrast to MON. It may be vulnerable (as the other two) to higher-order vagueness concerns, since it moves the border between non-gappy (i.e. contradictories) and gappy (i.e. contraries) down the series. Yet, since it is compatible with (many) non-classical accounts of vagueness and is supported by evidence, I think it is the best local principle available for explaining the role of soritical doublets. Thus, LI is an idealized constraint, but a highly motivated one.

To summarize, three first-order claims form the deep structure of the Sorites, explaining the components of the intuitive discursive form. I propose we accept only LI and treat TOL and MON as maybe having plausibility, but lacking necessity. The payoff is large, as we will see in the next section that LI replaces double negation elimination and helps us interchange the conditional and conjunctive propositional forms of the Sorites paradox, against Fine. I will also show that Fine's argument against the disjunctive form loses its force, since it relies on treating MON as a logical law, an implausible claim. Then, in Section 4 we briefly discuss the limitations of propositional logic in expressing the first-order principles of the Sorites paradox.

### 3. FOUR PROPOSITIONAL FORMS

Despite the dominating position of the conditional form, all four classical modes of inference have been discussed as advancing the propositional Sorites<sup>22</sup>:

(CS) Conditional, uses *modus ponens*:

If  $A_1$  then  $A_2$ ;  $A_1 \therefore A_2$ .

If  $A_2$  then  $A_3$ ;  $A_2 \therefore A_3$ .

etc.

(JS) Conjunctive

Not both  $A_1$  and not  $A_2$ ;  $A_1 \therefore A_2$ .

Not both  $A_2$  and not  $A_3$ ;  $A_2 \therefore A_3$ .

etc.

(DS) Disjunctive

not  $A_1$  or  $A_2$ ;  $A_1 \therefore A_2$ .

not  $A_2$  or  $A_3$ ;  $A_2 \therefore A_3$ .

etc.

(NS) Negative, uses *modus tollens*<sup>23</sup>

If  $A_2$  then  $A_3$ ; not  $A_3 \therefore$  not  $A_2$ .

If  $A_1$  then  $A_2$ ; not  $A_2 \therefore$  not  $A_1$ .

etc.

<sup>22</sup> Brian Weatherson, "True, truer, truest," *Philosophical Studies*, nr. 123 (online penultimate version), 2005, p. 11, <http://brian.weatherson.org/ttt.pdf>, accessed 21 December 2024.

<sup>23</sup> These Medieval Latin names can be translated as "affirming by affirming", "denying by affirming", "affirming by denying" and "denying by denying". They were recognized long before the rise of modern logic in the 19th century E.J. Lemmon, *Beginning Logic*, Chapman and Hall/CRC, 1971, p. 61.

We start the discussion with the negative form, since it is the least discussed variant. It will raise the issue of double-negation elimination, remarked by Fine in his discussion of his preferred conjunctive form. Then we will critically assess Fine's two distinct arguments against the conditional and disjunctive forms. First, I will show how LI unifies the conditional and conjunctive form. Secondly, I will show that Fine's argument against the disjunctive form relies on the implausible MON.

#### A) THE NEGATIVE FORM AND DOUBLE NEGATION ELIMINATION IN THE CONJUNCTIVE FORM

The negative form (i.e. modus tollens) is the least popular in philosophical literature. Perhaps because it shares the major premise(s) with the popular conditional form. But note that it easily converts an ascending to a descending Sorites or the converse. For example, you can create (CS.2) from (CS) above, while only replacing (CS.m) with a secondary minor premise:

(CS.2.m) A man of 100 cm is not tall.

Using modus tollens as the rule of inference, it will lead to the paradoxical:

(CS.2.C) A man of 200 cm is not tall.

As importantly, the negative form also helps convert a descending Sorites for one predicate to an ascending one for its complement. You can create (CS.3) from (CS) above, while replacing (CS.m) with:

(CS.3.m) A man of 100 cm is short

and replacing "tall" with "not short" in all of (CS.M.1- CS.M.100). Then by modus tollens and double negation elimination (DNE) you get:

(CS.3.C) A man of 200 cm is short.

Of course, DNE is controversial. Kit Fine notes<sup>24</sup> that in the conjunctive (JS) form above, the first intermediate conclusion is in fact  $\neg\neg A_2$ , and DNE is needed to get  $A_2$  for reuse with the next step. Although LI is strictly weaker than global interdefinability, it suffices for the local transitions needed here, replacing any need of DNE. In soritical reasoning, we will apply LI to replace  $\neg$ Tall with Short, eliminate the second conjunct, then re-apply LI to recover Tall:

<sup>24</sup> Kit Fine, "The possibility of vagueness," p. 3711.

(JS.2.r)  $a_1 \sim a_2, a_2 \sim a_3, a_3 \sim a_4 \dots$  – *soritical adjacency*  
 (JS.2.m) Tall( $a_1$ ) – *minor premise*  
 (JS.2.m.1) Not both Tall( $a_1$ ) and  $\neg$ Tall( $a_2$ ) – *first major premise*  
 (JS.2.m.1.s) Not both Tall( $a_1$ ) and Short( $a_2$ ) – *from (JS.2.m) and (JS.2.m.1) by LI*  
 (JS.2.C)  $\neg$ Short( $a_2$ ) – *intermediate conclusion by conjunctive syllogism*  
 (JS.2.m.2) Not both Tall( $a_2$ ) and  $\neg$ Tall( $a_3$ ) – *second major premise*  
 (JS.2.m.2.s) Not both  $\neg$ Short( $a_2$ ) and Short( $a_3$ ) – *from (JS.2.m.2) by LI twice*  
 (JS.2.C.2)  $\neg$ Short( $a_3$ ) – *by conjunctive syllogism from (JS.2.C) and (JS.2.m.2.s)*  
 ... and so on.

The paradox advances without DNE, once LI is granted, since the individual steps are always adjacent to one another, extending the interdefinability. LI is weak, but sufficient to let the paradox advance even intuitionistically.

## B) THE CONDITIONAL FORM AND METALINGUISTIC NEGATION

We just saw LI salvaging the conjunctive form, being as strong as DNE in local context. Now, Fine denies the plausibility of the conditional form of the major premise(s), thus of both the conditional and negative forms of the paradox. I think that both the conditional and conjunctive forms are motivated by the interplay of TOL and LI, unifying Fine's divergent treatment of the two.

Fine's argument starts by comparing what he takes the conditional and conjunctive forms to express, conditions he calls, respectively, "*Tolerance*<sup>\*25</sup>" (a different notion than Wright's principle of tolerance and our TOL above) and "*Cut-Off*":

Tolerance\* permits us to make the transition from the truth of the antecedent claim (that Left is bald) to the truth of the consequent claim (that Right is not bald) [sic]<sup>26</sup>, while Cut-Off forbids us from making the transition from the truth of the antecedent claim to the falsity of the consequent claim. But to say that we are forbidden (given the truth of the antecedent claim) to assert the falsity of the consequent claim is not to say that we are permitted to assert the truth of the consequent claim unless we also take for granted that the consequent claim is either true or false.<sup>27</sup>

Clearly, Fine intends a non-material reading of the conditional. Asserting the conditional major premise(s), he says, is saying that the consequent can be asserted when the antecedent is true, which Fine thinks unmotivated. While the conjunctive major premise(s) says only that the consequent must not be denied in that situation,

<sup>25</sup> I will use the \* to identify Fine's principle, including inside Fine's quote.

<sup>26</sup> This is a typo; it is of course to be read as 'that Right is bald'.

<sup>27</sup> Kit Fine, "The possibility of vagueness," p. 3713.

which he thinks motivated. After limiting his discussion to the formulation of major premise(s) themselves, he will go on to distinguish the semantics of these two rules of inference, seeing conjunctive syllogism fail while modus ponens keeps its validity.

Note that Fine's gloss of the conjunctive premise does more than restate  $\neg(A \wedge \neg B)$  as a metalinguistic interdiction. Fine appears to move from the external denial of the conjunction, together with the truth of the antecedent, to the external denial of the consequent's falsity. But if the outer negation is genuinely metalinguistic, this transition is not purely formal. It requires a principle governing the descent of external negation under a settled conjunct. And if the outer negation was not metalinguistic, then it is unclear how such a negation showed up at B.

I think Fine's reduction of outer negation should satisfy two conditions in order to be principled. Firstly, the move from  $\neg\text{Ext}(\varphi \wedge \psi)$  to  $\neg\text{Ext}(\psi)$ , once  $\varphi$  is assumed as true, must reflect a general rule available when part of the scope of the external negation is settled. Secondly, the rule must respect negation hierarchy uniformly across the forms of the paradox<sup>28</sup>. One cannot treat external negation as a genuine operator when helpful, but as merely informal gloss otherwise.

But we saw that in adjacent soritical cases LI licenses replacing  $\neg B$  with its contrary. Then, if B is Tall(Right), then  $\neg B$  is locally Short(Right). The scope of the reduced external negation will no longer be a falsity claim as in the case of DNE, but a positive predicate, namely "short". If Fine's original reduction pushes outer negation onto what remains unsettled, the same hierarchical logic will keep descending if the scope is settled, i.e. a positive atomic predicate. Therefore, the metalinguistic interdiction should become standard negation after LI is applied. Otherwise, the operator  $\neg\text{Ext}$  is used ad hoc, only descending when it favors Fine's contrast.

The conclusion is that if Fine's reduction is granted as a general hierarchical rule, and LI is granted as well, the conjunctive premise yields denial of the contrary and, by LI again, the original consequent. Even if Fine uses "given that..." which usually indicates the conditional, I do not assert that the conjunctive and conditional forms become identical. But Fine can no longer claim a motivational advantage for the conjunctive form. And that is as it should be, since LI gives an account of the way speakers switch soritical doublets across any propositional forms.

For a derivation, let  $\neg\text{Ext}$  be external negation, and let  $cB$  be the relevant contrary of B.

(F) From  $\neg\text{Ext}(\varphi \wedge \psi)$  and  $T(\varphi)$ , infer  $\neg\text{Ext}(\psi)$ .

This is our reconstruction of Fine's reduction.

(H) From  $\neg\text{Ext}(P)$ , where P is a positive atomic claim, infer  $\neg P$ .

<sup>28</sup> That may be the case internally for Fine, e.g. one may reject DNE by reading double negation as  $\neg\text{Ext}(\neg\varphi)$ .

This is a principled extension of (F) once external negation is applied to a single positive content.

- (F.1)  $\neg\text{Ext}(A \wedge \neg B)$  – *premise*
- (F.2)  $T(A)$  – *premise*
- (F.3)  $\neg\text{Ext}(\neg B)$  – *from (F.1), (F.2) by (F)*
- (F.4)  $\neg\text{Ext}(cB)$  – *from (F.3) by LI*
- (F.5)  $\neg cB$  – *from (F.4) by (H)*
- (F.6)  $B$  – *from (F.5) by LI*

Therefore, given  $T(A)$ , we obtain  $B$ . If truth is eliminable, this recovers the conditional form.

Against the possible objection that this over-interprets Fine’s words, note that either his reduction is principled, or it is ad hoc. If it is principled, then it should generalize as a treatment of hierarchical negation, and then the derivation is legitimate. If it is ad hoc, then it cannot ground the claim that the conjunctive form is uniquely well-motivated.

### C) THE DISJUNCTIVE FORM AND LAW OF EXCLUDED MIDDLE

Fine also attacks the motivation of the disjunctive form of the paradox. He proceeds in four steps<sup>29</sup>:

- (KF.D.1) The Sorites is about a kind of indeterminateness of predicates.
- (KF.D.2) Indeterminateness means at least that the law of excluded middle (*LEM*) is not plausible.
- (KF.D.3) Saying “Either a man of 200 cm is not tall or one of 199 cm is so” is an even stronger statement than *LEM* for the first step predicate, that is “Either a man of 200 cm is not tall or a man of 200 cm is tall”. The reason is that the first disjunct is identical and the second disjunct of the former statement is stronger than the second disjunct of the latter, because by the principle that a man of more cm will be tall provided another of less cm is, it implies it.
- (KF.D.C) Therefore, the disjunctive soritical major premise(s) is not plausible.

First, the third step assumes that all tall people have more cm than all short (not tall) people, namely our *MON*. But, as argued above, it is not a logical law, and it is of lower reliability than *LI*, since there are *prima facie* counterexamples. Of course, the disjunctive statement may be statistically stronger, namely that monotonicity may be of a higher likelihood than *LEM*, but deductively it is just as strong.

<sup>29</sup> Kit Fine, “The possibility of vagueness,” p. 3712.

Secondly, LEM is used in many ordinary contexts involving vague predicates, especially when the speaker is forced to decide. Near the end, LEM should not cause any trouble to Fine. Since men of 200 cm are tall, we can infer that they are either tall or short. Might LEM then be a problem at doubtful cases of in-between heights, as has been often claimed<sup>30</sup>? First, suppose John believes in Fine's Cutoff principle, expressed as:

(NoCutoff) There is no way that being one cm apart from a tall person is compatible with shortness.<sup>31</sup>

And suppose Mary believes in:

(TallRich) There is no way that being tall on Wall Street is compatible with poverty.

People such as Mary often reason disjunctively with soritical predicates:

(TallRich.M) On Wall Street, you are either not tall, or you are rich.

And if (TallRich.M) is a natural expression of belief in (TallRich), a belief in (NoCutoff) leads to:

(NoCutoff.M) Of people one cm apart, either one is not tall, or the other is tall as well.

LEM is indeed rare in common speech. People do not say unprompted that right angles are over 90 degrees or not, whether the predicate is precise or not. But it is exactly when principles force them to choose in doubtful cases, i.e. when common language reasoning gets going, that we use it. Love is a very doubtful notion, but "You either love me or you do not" may be the commonest LEM utterance there is. Since both LEM and disjunctive syllogisms are commonly used with vague predicates in doubtful cases, what basis would someone have to not accept (NoCutoff.M) given (NoCutoff)?

Thus, if we don't accept MON – and we should not –, the disjunctive form of the Sorites rests solely upon the plausibility of LEM in vague contexts, where reasonable everyday examples have been supplied.

<sup>30</sup> "The failure of excluded middle may seem natural enough in borderline cases" Timothy Williamson, *Vagueness*, p. 118.

<sup>31</sup> Fine's formulation is "If two cases are sufficiently alike then it is not the case that the first is bald and the second is not". Kit Fine, "The possibility of vagueness," p. 3713.

#### 4. THE LIMITS OF THE PROPOSITIONAL FORM

The three claims extracted in Section 2 are not logical laws. TOL extends the distribution of “tall” for all men of less cm than a tall man. MON imposes a monotonic distribution of “tall” for all men of more cm than a tall man. LI inter-defines “tall” and “short” at any one step away from any short or tall case.

Since it is the only one with empirical justification, I use LI in this analysis of the Sorites paradox. But all of them are first-order, can they really be captured propositionally? The propositional form expresses the crucial relation (discrete “ $\leq$ ” or dense “ $\sim$ ”) as the succession of steps e.g. in (CS). Otherwise, LI has no trace in the propositional transcriptions of the paradox. As for MON, it is absent from the transcription, although it is present in many non-classical treatments of vagueness at the level of semantics, e.g. supervaluationism enforces it as penumbral connections. Regarding TOL, it is instantiated in the individual propositional steps. Since all cases from 200 cm to 100 cm are covered by the relation between each antecedent and consequent, is the propositional form enough? Fine wrote: “Let us focus on the sentential rather than the quantificational versions of the soritic arguments, since I doubt that the quantificational versions give rise to any essentially new issues”<sup>32</sup>.

There are three problems with such an approach.

Firstly, the propositional form (or for that matter, the inductive form) does not express some quasi-logical relationships between the arbitrary parameters. Those are the precise initial and final quantity (e.g. 200 and 100), the precise soritical step (e.g. 1) and the precise number of steps to get between them (e.g. 100). As discussed, neither the propositional nor the inductive form expresses the *direction* of application. And the argument may possibly be investigated in different first-order versions which are subtly but importantly different. Without discussing variants here, saying “*if some man of 187 cm is tall, there is a man of 186 cm which is also tall*” is different from saying “*if all men of 187 cm are tall, then all men of 186 cm are tall*”, or that “*for this specific person, if only their height changed, the predicate ‘tall’ would still apply*”.

Secondly, TOL for example states a relation between predicates i.e. sets of elements or units of meaning, while (CS.M.1) states a relation between the truth values of two sentences. Propositional treatments of the Sorites give up expressing relations such as that between the start point, end point and numerical step in the object language, hence the debate is confined to the metatheory: the semantics or proof theory embed MON and weaken TOL, hiding them from the object language.

And thirdly, the propositional forms of the Sorites are *too generic*. They are shared with many other sound and unsound arguments, for example this adaptation of a well-known argument of Rachels’:<sup>33</sup>

<sup>32</sup> *Ibidem*, p. 3710.

<sup>33</sup> Stuart Rachels, “Counterexamples to the transitivity of *better than*,” *Australasian Journal of Philosophy*, vol. 76, nr. 1, March 1998.

(R.m) 5000 years of slightly less ecstasy are preferable to 50 years of ecstasy.	A
(R.M.1) If 5000 years of slightly less ecstasy are preferable to 50 years of ecstasy, $5 * 10^5$ years of slightly less ecstasy are preferable to 50 years of ecstasy	$A \supset M_1$
(R.M.2) If $5 * 10^5$ years of slightly less ecstasy are preferable to 50 years of ecstasy, $5 * 10^7$ years of slightly less ecstasy are preferable to 50 years of ecstasy.	$M_1 \supset M_2$
...	
(R.M.23) If $5 * 10^{49}$ years of slightly less ecstasy are preferable to 50 years of ecstasy, $5 * 10^{51}$ years of slightly less ecstasy are preferable to 50 years of ecstasy.	$M_{23} \supset B$
(R.C) $5 * 10^{51}$ years of slightly less ecstasy are preferable to 50 years of ecstasy.	B

Of course, Rachels' argument is against the transitivity of preference, at least for pleasures and pains, since it suggests that  $5 * 10^{51}$  years are simply too many to be preferred to 50 years<sup>34</sup>. Yet it can be cast in the same logical form as a propositional Sorites. In fact, any chained argument (i.e. transitivity-related) can be cast in this form, which shows that the propositional forms of the Sorites are not expressive enough.

The conclusion is that the Sorites paradox lies above propositional level, investigating such claims as TOL, MON and LI. Since the various propositional forms of the Sorites paradox are surface-level expressions of them, the validity of these propositional forms depends on first-order and semantic assumptions, rather than solely logical form. The conjunctive form is not privileged, as Fine claims, just as the conditional form is not privileged, as previously implied by many fuzzy logicians. While most of the workings of the Sorites paradox are first-order, if we want to capture also the seemingly logical relations between the elements of the paradox such as that for any start and end points, there are a finite number of connecting steps, we may need higher-order tools<sup>35</sup>.

<sup>34</sup> *Ibidem*, p. 76.

<sup>35</sup> This paper substantially revises and develops material from Chapters 1 and 2 of Marian Călborean, "The nature and logic of vagueness" (PhD Thesis, University of Bucharest, 2020).