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LOGICISM, FREGE AND THE LOGICAL COMPOSITION OF INFORMATIVE IDENTITIES

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Abstract: Informative, non-trivial identities (like the definition of number) form the core of Frege's logicist program. This is why it is important to be able to dissipate any impression that there might be something paradoxical about them. However, it is easy to have such an impression : informative identities seem to be able to have either correctness (because the two elements involved in the identity really are identical) or informativeness (because the two elements involved in the identity are not exactly identical). This problem is sometimes called 'the paradox of analysis.' The first step in the direction of a Fregean solution to the paradox is, I argue, the elucidation of the status of 'fruitful definitions' as informative identities in Frege.

Keywords: Frege, fruitful, definition, informative identities, analysis

The entire logicist program relies on the assumption that there can be such things as informative identities, i.e., definitions, analyses, equations that are both analytic and informative. However, this lands the program in the middle of a problem known as 'the paradox of analysis'[1]: how can an identity statement be both correct (and therefore the *analysans* be the same as the *analysandum*, justifying the identity sign) and informative? It would seem that, in order to be informative the *analysandum* has to be somehow significantly different from the *analysans*. And this is the exact same conundrum one meets when having to do with an identity statement that is supposed to be both analytic and informative.

The paradox of analysis has a long and respectable tradition and I will not attempt to give here a way out of it. Rather, the purpose of this essay is more

[1] The paradox of analysis appears under this specific name especially in connection with G. E. Moore, namely when C. H. Langford questions Moore's notion and method of analysis: 'It is indeed possible to deny that analysis can be a significant philosophical or logical procedure. This is possible, in particular, on the ground of the so called paradox of analysis, which may be formulated as follows. Let us call what is to be analyzed the *analysandum*, and let us call that which does the analyzing, the *analysans*. The analysis then states an appropriate relation of equivalence between *analysandum* and the *analysans*. And the paradox of analysis is to the effect that, if the verbal expression representing the *analysandum* has the same meaning as the verbal expression representing the *analysans*, the analysis states a bare identity and is trivial; but if the two verbal expressions do not have the same meaning, the analysis is incorrect.' - in C.H. Langford, "Moore's Notion of Analysis", in *The Philosophy of G.E. Moore*, edited by Paul Arthur Schilpp, New York: Tudor Publishing Company, 1952. p. 322.

modest: to clear a path towards a Fregean kind of solution and to clarify a couple of points of Fregean exegesis. Specifically, I want to discuss what it meant for Frege to give a 'fruitful definition' and if this can be considered to be the same with an informative identity statement.

Frege's own formulation of the paradox of analysis can be found in his review to Edmund Husserl's *Philosophie der Arithmetik*[2], where the dilemma is presented as a Husserlian objection:

> If words and combinations of words mean ideas, then for any two of them there are only two possibilities: either they designate the same idea or they designate different ideas. In the former case, it is pointless to equate them by means of a definition: this is an 'obvious circle'; in the latter case is wrong. These are also the objections the author raises [i.e. Husserl], one of them regularly. A definition is also incapable of analyzing the sense, for the analyzed sense is just not the original one. In using the word to be explained, I either think clearly everything I think when I use the defining expression: we then have the 'obvious circle'; or the defining expression has a more richly articulated sense, in which case, I do not think the same thing in using it as I do in using the word to be explained: the definition is then wrong.[3]

One more trait of the discussion emerges from the above quote by its reference to definitions: the paradox of analysis places under scrutiny definitions, analyses, explanations, i.e. statements that may be subsumed under the general heading 'identity statements'. The dilemma stems from (at least apparently) conflicting requirements: in order to be correct, the right and the left side of the identity statement should have something in common; on the other hand, in order to be non-trivial, the two sides of the identity must also incorporate significant differences.

Possible ways out of the paradox will usually distinguish between two elements of the identity: one that stays the same on both sides and one element which differs from one side to another of the identity. Obviously, the one element which stays the same will account for the correctness of the identity relation and the relevant difference between terms will account for the informativeness of the identity, i.e. for the new piece of information obtained by acknowledging the identity. A Fregean solution, I believe, would have to rely on the famous same reference-different sense kind of explanation[4], but this is the topic for another

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[2] In 1891 Husserl's *Philosophie der Arithmetik* was published; several points from Frege's *Grundlagen* (1884) are criticized here, including the crucial definition of number by means of one-to-one correlation. Frege's reply comes in 1894, when his review of Husserl's *Philosophie der Arithmetik* was published in *Zeitschrift fur Philosophie und phil. Kritik*, vol. 103.

[3] G. Frege, 'Review: Husserl, Philosophy of Arithmetic' in *Collected Papers on Mathematics, Logic and Philosophy*, p. 199.

[4] This is a rather controversial point with many complicated adjacent problems to be solved, like the compositionality of thoughts, the status of "content"(*Inhalt*) in Frege and so on. See Michael Dummett and Michael Beaney for a different opinion about a possible Frgean solution to the paradox of analysis.

essay. What I want to establish here, for the time being, is that Frege does think that there are such things as informative identities and that some of them are exactly what he himself has called sometimes ‘fruitful definitions’. This is an important step, I believe, towards explaining how definitions – central to the logicist project (like the definition of number) – can be seen as both analytic and informative or ‘fruitful’. In other words, the bigger project is to show that in order to explain how informative identities are possible in a Fregean framework, one has to adopt a different sense-same reference solution. But the first step towards this bigger project is this, rather exegetical point, namely to show that what Frege calls ‘fruitful definitions’ (like the definition of number) are a kind of informative identities.

1. FRUITFULNESS AND INFORMATIVENESS IN FREGE'S TEXTS – THE PROBLEM

The term ‘fruitful’ is used by Frege in respect to certain definitions, procedures and concepts only in his early period, namely in ‘Boole’s logical Calculus and the Concept-script’ and in *Grundlagen*. But this term is not used any more after *Grundlagen*; a different expression is used, namely ‘informative’ or ‘far greater cognitive value’[5]. One may interpret ‘fruitful’ as different from ‘informative’ in order to dissolve a certain tension in Frege’s account of definitions. Commentators often distinguish them, interpreting ‘fruitful’ as ‘useful for making logical connection and inferences’ and ‘informative’ as ‘giving information about the world, about reality’.

Therefore, in a plausible interpretation of the Fregean text, the situations where ‘to be fruitful’ is applied are considered different from the situations where Frege presents his famous informative identities.

Authors prefer to distinguish the two situations mainly because it offers a solution to an important problem in Fregean exegesis, namely it brings coherence to Frege’s stand in respect to definitions[6]. Frege is famous for his oscillations regarding the role and importance of definitions. On the one hand, in early writings, he emphatically affirms that at least some definitions (actually, key definitions for his logicist enterprise) should be regarded as ‘fruitful’. For example, in *Grundlagen*, he states:

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[5] See for example “Notes to Ludwig Darmstaedter”, written in 1919, in *Posthumous Writings* p. 255.

[6] For the connection between ‘fruitfulness’ and inferential role of definitions, see Tappenden Jamie, “Extending Knowledge and ‘Fruitful Concepts’: Fregean Themes in the Foundations of Mathematics”, in *Nous,* Vol. 29, No 4, (1995) and for a difference between an inferential role (supposedly endorsed by Russell and Russellians) of *Sinn* see Taschek William, “Frege’s Puzzle, Sense, and Information Content” reprinted in volume VI of *Gottlob Frege: Critical Assessments of Leading Philosophers,* edited by Michael Beaney and Erich Reck, London, Routledge, 2005.

> The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with all the rest.[7]

On the other hand, he equally emphatically states in a late writing:

> And even if, what a definition has stipulated is subsequently expressed as an assertion, still its epistemic value is no greater than that of an example of the law of identity a=a. *By defining no knowledge is engendered*. (…) No definition extends our knowledge. It is only a means for collecting manifold content into a brief word or sign, thereby making it easier for us to handle. This and this alone is the use of definitions in mathematics.[8]

One way out of the difficulty is to simply say that, in time, Frege changed his mind in respect to definitions, i.e. in the beginning he considered them as closer to informative identities, of the type a=b, and later he considered them as being rather closer to the uninformative identities, of the type a=a.

Another way out would be to distinguish between 'informative identities' (i.e. the famous kind of identities that have, according to Frege, 'far greater cognitive value') and 'fruitful' identities, and in this way to be able to say, without contradiction, that some definitions are fruitful, but no definition is informative. On this interpretation, 'fruitful' would mean 'useful for making logical connection and inferences' and 'informative' would be interpreted as 'giving information about the world, about reality'. The status of definitions depicted in this way is, indeed, a plausible one: some definitions would be useful tools in an inference, but they would give no information about reality.

In short, Frege does not seem to have a straight answer to the question: are definitions informative or not? The definitions obtained by stipulation are not, of course, but Frege presents other kinds of definitions about which he claims that they are 'fruitful' therefore they could be qualified as (at least) non-trivial. Frege's examples from *Grundlagen* are his own definitions of the concept of number and of number 0 or 1, and the definition of the continuity of a function; other examples are brought in 'Boole's logical Calculus and the Concept-script' (of 'congruence of two numbers with respect to a modulus'). It might be objected that these are 'fruitful' but not necessarily 'informative'. Still, Frege seems to make a distinction between two kinds of definitions, one that are more informative than the others. both before and after the distinction between *Sinn* and *Bedeutung*. For example, in *Begriffsschrift* when he speaks about 'equality of content' (i.e. identities), he also makes the distinction between a mere abbreviation and the situation when 'different names for the same content are not always just a trivial matter of formulation; if they go along with different ways of determining content, they are

[7] G. Frege, *Foundations of Arithmetic,* p. 100.

[8] G. Frege, "On the Foundations of Geometry" in *Collected Papers,* p. 274.

relevant to the essential nature of the case.'[9] Besides, in the earlier quoted review of Husserl's *Philosophie der Arithmetik,* he clearly states the possible case of definitions for which the *definien*s and the *definiendum* have the same *Bedeutung* but different *Sinn.*

One may think that a solution to this ambiguity can be brought by Frege's distinction between kinds of definitions from *Logic in Mathematics*. But as I will argue in the next section, that distinction is not a solution to the problem of fruitful definitions because it was not intended to be a solution to this problem. Frege's aim was completely different.

2. CONSTRUCTIVE AND ANALYTIC DEFINITIONS IN FREGE – NOT A SOLUTION

The status of definitions for Frege is an ambiguous one, or, at least, it is ambiguous until the clarifications brought in *Logic in Mathematics*, some authors maintain. The ambiguity comes from the fact that, on the one hand, definitions, seen as stipulations, are identities that have in common both the *Sinn* and the *Bedeutung* of the terms flanking the identity sign; on the other hand, according to Frege, there are some identities that can be informative, like the famous 'Morning Star = Evening Star' from his 'Über Sinn und Bedeutung', i.e., identities that are non-trivial, of the form a=b. The distinctive feature of such non-trivial identities is that the two sides of the identity have the same *Bedeutung* but not the same *Sinn.* One might reasonably presume that Frege's definitions from *Grundlagen* (i.e., the definitions of the number 1 or 0 or the definition for 'number' in general) are considered by Frege himself as non-trivial identities and, therefore, according to Frege's suggestion, to have only their *Bedeutung* in common. There is an oscillation between a 'same sense' account and a 'different sense' account for the case of definitions, most probably related with the case of special identities (like the definition of number) where more than sameness of *Bedeutung* is required, but also less than sameness of sense. It would seem that this tension between 'same sense'/'different sense' requirements for definitions would disappear if we could distinguish between two kinds of definitions: some that require only sameness of *Bedeutung* (and they would be the non-trivial ones) and some that require sameness of both *Sinn* and *Bedeutung* (and they would be the trivial, stipulative ones). And this seems to be the path taken by Frege in *Logic in Mathematics*, where he distinguishes between 'constructive' (*aufbauende*) definitions and 'analytic' (*zerlegende*) definitions. But we might expect that the outcome is not so simple. The constructive definitions are, indeed, a simple case of stipulative definitions, when a sign (or group of signs) is attached as a *definiens* to a *definiendum*. But in the case of analytic definitions, (i.e., the ones given for a sign with a 'long,

[9] See p. 12 in *Translations From Philosophical Writings of Gottlob Frege*, edited by Peter Geach and Max Black.

established use') Frege distinguishes two cases: the case when the sign to be defined has obviously the same sense as the *definiens*; and the case when the sign to be defined does not have obviously the same sense as the *definiens*. In the first case, Frege states, the result is rather an axiom than a definition.

However, the second case is the interesting one because this is the case when the sense of the *definiendum* may be different from the sense of the *definiens.* Frege's brief recommendation in this case leaves no place for further pondering the question: The sign already in use (the *definiendum*) is to be replaced by a new sign by stipulation and in this way we avoid the incertitude and the problem of deciding if the old sign really has the same sense or not with the proposed definition for it (i.e., with the *definiens*). In other words, this case is reduced to the case of 'constructive' definitions.

I think that Frege's reduction of analytic definitions to constructive ones by making them to depend also on stipulations is meant to solve a completely different problem. I believe that the worry here is one often times expressed by Frege, namely that a scientific system must make clear from the beginning every element that stays at its basis so that no tacit supposition would support it. In this way, if something goes wrong in the system, the mistake can be traced back to the primitive assumptions. The imperative here is that, considering the stipulations as assumed risks in constructing the system, it must be clear *which exactly* the assumed risks are. This is why, if we are not sure that the senses between the old sign and the new expression coincide, we must *make clear* that we are not sure, but we *take a risk* and so we make an *explicit stipulation*. It is exactly what Frege does in the beginning of his *Grundgesetze* when he explicitly states the danger coming from Axiom V, and when he mentions that this is the *only place* where the danger could come from. The problem that he addresses here by making the distinction between the two kinds of definitions, I believe it to be caused rather by his famous dispute with the formalist positions (with Hilbert, especially). Frege often states in his disputes regarding the foundations of geometry his opinion that basic notions like 'point' or 'line'[10] cannot be supposed to be defined simply by their usage, while what really happens is that their old senses, from the common language, are surreptitiously used. His prescription is always the same for this situation: we should not clandestinely use the old sense, established by the previous usage; but we should say overtly if we are going to use it and if we suppose that it *is the same or not*. A word or an expression cannot be defined simply by its usage in a system – true or not as a thesis *per se*, I believe that this was, rather, Frege's thesis defended by distinguishing the two situations when defining something.

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[10] In 'On the Foundations of Geometry: Second Series' (1906), Frege states: "We are easily misled by the fact that the words 'point', 'line', etc. have already been in use for a long time. But just imagine the old words completely replaced by new ones especially invented for this purpose, so that no sense is as yet associated with them. And now ask whether everyone would understand the Hilbertian axioms and definitions in this form. It would amount to pure guess-work. Some would perhaps not be able to guess anything at all; some this, other that.' p. 301.

This different view[11] upon Frege's purpose in 'Logic in Mathematics' is supported, I believe, by two arguments that can be made about the text. Firstly, in the same text, both Hilbert (for the fluctuating sense of 'axiom' in his *Grundlagen der Geometrie*) and Weierstrass (for resting his theory on the tacit understanding for 'number', even though his explicit definition is different from the old, implicit sense) are criticized for the above mentioned procedure: using in their systematical writings about mathematics terms already in use and so non-explicitly employing their old sense.

Secondly, the overall aim of both the theory about the definitions and the critiques brought to Hilbert and Weierstrass is explicitly stated together with the worry that has generated them. Namely, all the norms in using signs and definitions are given with the purpose of constructing a genuine system in arithmetic:

> As a science develops, a certain system may prove no longer to be adequate, not because parts of it are recognized to be false, but because we wish, quite rightly, to assemble a large mass of detail under a more comprehensive point of view in order to obtain greater commend of the material and a simpler way of formulating things. (...) In this way it can happen that sentences which meant the True before the shift, mean the False afterwards. Former proofs lose their cogency. Everything begins to totter. We shall avoid all these disasters if, instead of providing old expressions or signs with new meanings, we introduce wholly new signs for the new concepts we have introduced. But this is not what usually happens; we continue instead to use the same signs. If we have a system with definitions that are of some use and aren't merely there as ornaments, but are taken seriously, this puts a stop to such shifts taking place. We have then an alternative: either to introduce completely new designations for the new concepts, relations, functions which occur, or to abandon the system so as to erect a new one. In fact we have at present no system in arithmetic. All we have are movements in that direction. Definitions are set up,

[11] This is an indirect argument, but one may notice that the distinction in question, between analytic and constructive definitions does not appear as such in Carnap's notes taken at Frege's course. This is, of course, only an indirect testimony, but it can show that there was no emphasis placed on distinguishing two kinds of definitions. In the course notes only the term 'definition' as such appears and the prescription of the stipulation for a new sign in case we are not certain that the analysis does not correspond to the sense of the old sign. As it is little known, I will quote the entire passage covering the issue from Carnap's notes: 'Definitions don't just help to construct, but also to analyze what is complex, e.g., in order to reduce the number of axioms. Such an analysis cannot be proved right; one can only feel that one has hit the nail on the head, and it can prove itself fruitful. More precisely: we construct the system again by using the result of the analysis. Occasionally what is fixed in a definition is the sense of a sign that has already been in use for a while. One cannot prove that, though; it has to be evident; it isn't an arbitrary stipulation, but an axiom. Let A be the old sign; let's assume that a certain complex sign coincides in sense with A. If we don't know that for sure, we proceed as it follows: We stipulate arbitrarily that B is to have the same sense of the complex sign. If the 1st definition was correct, then the sense of A has to coincide with that of B. We avoid the sign A and reconstruct the whole system by using B. If that reconstruction succeeds, we can, for pragmatic reasons, also introduce the sign A again; we just have to regard it as newly introduced, as if it hadn't had a sense before the definition.' - in *Frege's Lectures on Logic: Carnap's Student Notes, 1910–1914*, Rech, E., and Awodey, S. (translators and editors), Chicago and LaSalle, IL, Open Court, 2004.

> but it doesn't so much as enter the author's head to take them seriously and to hold himself bound by them. So there is nothing to place any check on our associating, quite unwittingly, a different meaning with a sign or word.[12]

Therefore, I think it is quite clear that the distinction between constructive and analytic definitions was meant to solve a completely different problem than the problem of informative definitions. The problem of the status of definitions *per se* is not the main problem here, but the distinction between 'fruitful' and 'informative' is a problem of major importance. My perspective relies on a different interpretation of the Fregean text, namely on the exegetic hypothesis that both terms 'fruitful' and 'informative' refer to the same kind of informative identities. Consequently, in my interpretation the two problematic terms (i.e., 'fruitful' and 'informative') are not differentiated in the manner described above; on the contrary, a lot rests on their solidarity. In what follows I will bring several arguments in favor of this departure from the previously described interpretation (of different meanings for the two terms).

3. THE COMMON SIDE OF FRUITFULNESS AND INFORMATIVENESS

Frege details his usage of the term 'fruitful' in two main passages. These passages will be discussed in detail in the next subsection. The main concern here is to bring forth a possible interpretation of the Fregean text that would underline the solidarity between Frege's 'fruitfulness' and 'informativeness'.

Fruitfulness appears illustrated by the metaphor of 'drawing boundaries' in 'Boole's logical Calculus and the Concept- script', accompanied by actual drawings of diagrams:

> If we look at what we have in the diagrams we notice that in both cases the boundary of the concept whether it is one formed by logical multiplication or addition is made up of parts of the boundaries of the concepts already given. This holds for any concept formation that can be represented by the Boolean notation.
> This feature of the diagrams is naturally an expression of something inherent in the situation itself, but which is hard to express without recourse to imagery. In this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines. These really already contain the new concepts: all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way. It is the fact that attention is principally given to this sort of formation of new concepts from old ones, while other more fruitful ones are neglected which surely is responsible for the impression one easily gets in logic that for all our to-ing and fro-ing we

[12] G. Frege, 'Logic in Mathematics,' in *Posthumous Writings,* p. 242.

> never really leave the same spot. (...). If we compare what we have here with the definitions contained in our examples, of the continuity of a function and of a limit, and again that of following a series which I gave in § 26 of my *Begriffsschrift,* we see that there's no question there of using the boundary lines of concepts we already have to form the boundaries of the new ones. Rather, *totally new boundary lines* are drawn by such definitions – *and these are the scientifically fruitful ones*. Here too, we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways by means of the signs for generality, negation and the conditional.[13]

The metaphor of boundaries and the distinction between 'to use the lines already given' and 'drawing boundary lines that were not previously given at all' also appear in a famous passage in paragraph 88 from *Foundations:*

> He [Kant] seems to think of concepts as defined by giving a simple list of characteristics in no special order; but of all ways of forming concepts, that is one of the least fruitful. (...) The same is true of the really fruitful definitions in mathematics, such as that of the continuity of a function. What we find in these is not a simple list of characteristics; every element in the definition is intimately, I might almost say organically, connected with all the rest. A geometrical illustration will make the distinction clear to intuition. If we represent the concepts (or their extensions of these) by figures or areas in a plane, then the concept defined by a simple list of characteristics corresponds to the area common to all the areas representing the defining characteristics; it is enclosed by segments of their boundary lines. With a definition like this, therefore, what we do – in terms of our illustration – is to use the lines already given in a new way for the purpose of demarcating an area. Nothing essentially new, however, emerges in the process. But the more fruitful type of definition is a matter of drawing boundary lines that were not previously given at all. What we shall be able to infer from it, cannot be inspected in advance; here, we are not simply taking out of the box again what we have just put into it. The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic. The truth is that they are contained in the definitions, but as plants are contained in their seeds, not as beams are contained in a house. [14]

It can be noticed from the above quote that the 'fruitfulness' is not completely absent from the first kind of definitions: the problem is not that there is *nothing new* (actually there is 'a new way of demarcation' mentioned); the problem is that there is nothing *essentially new (*the kind of definition proposed is just 'more fruitful' than the other).

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[13] G. Frege, "Boole's logical Calculus and the Concept-script", in *Posthumous Writings*, pp. 33-34.

[14] G. Frege, *Foundations of Arithmetic*, p 100–101, paragraph 88.

This is a passage reinforcing the tenets from the previous one. It is clear from these two passages that, according to Frege, there are two kinds of definitions: one kind more fruitful than the other. But the more fruitful kind has different senses for *definiendum* and *definiens*, i.e., the *definiendum* and *definiens* are made up of different 'boundaries' (the different boundaries being an analogy for different characteristics). The less fruitful kind is the one in which 'lines that are already there', the network of lines that 'really already contain the new concepts'.

To compare the two, in short:

'Less fruitful' method - previously given lines	'More fruitful' method - new lines
§ 88 in *Foundations*	
– a simple list of characteristics –'to use the lines already given in a new way for the purpose of demarcating an area' – 'nothing *essentially new* (...) emerges in the process' – 'as beams are contained in a house'	– elements organically connected – 'drawing boundary lines that were not previously given at all' – 'not simply taking out of the box again what we have just put into it' = 'what we shall be able to infer from it, cannot be inspected in advance' – 'as plants are contained in their seeds'
'Boole's logical Calculus and the Concept-script'	
– 'the boundary of the concept (...) is made up of parts of the boundaries of the concepts already given' – 'in this sort of concept formation, one must, then, assume as given a system of concepts, or speaking metaphorically, a network of lines' – 'these really already contain the new concepts : all one has to do is to use the lines that are already there to demarcate complete surface areas in a new way' – 'using the boundary lines of concepts we already have to form the boundaries of the new ones'	– '[r]ather, totally new boundary lines are drawn by such definitions – and these are the scientifically fruitful ones' – '[h]ere too, we use old concepts to construct new ones, but in so doing we combine the old ones together in a variety of ways'

The above table shows that both methods of forming concepts rely on 'using old concepts to construct the new ones'. The difference between them consists in the fact that in one case (the less fruitful method) the 'new' concept will have each part of its boundaries coinciding with some parts of the boundaries delimiting the old concepts; in the other case (the more fruitful method) the new concept will have its boundaries formed out of lines that were not present before as boundaries of the old concepts.

It becomes quite clear that on the less fruitful side the 'old' and the 'new' concepts share the 'same boundaries' or the 'same lines', i.e., the division is made along the same given lines. And actually, the constructed concepts are 'new' only because the old lines were used in a different manner. But a network of fixed, given lines results in always demarcating the same units of division, i.e., same parts. In other words, keeping the same lines results in always having the same parts; their configuration cannot change because the demarcation lines cannot change (the only change that can occur consists in a different combination of the parts). This is a process of arriving from ‘old’ to ‘new’ by minimum changes: neither the lines or the whole changes, only the position of the given parts (i.e., of the areas delimited by fixed lines) taken in different combinations, combinations that can be calculated in advance. This is the situation characterized by combinatorial calculus. By contrast, the other situation presented above (the ‘more fruitful’ method) is the one in which ‘we do not simply take out of the box again what we have just put into it’ and where the results cannot be calculated in advance: ‘what we shall be able to infer from it, cannot be inspected in advance’. In this case we are ‘drawing boundary lines that were not previously given at all’; by doing this we involve also a change in the configuration of the division, i.e., we are changing the boundaries of the ‘parts’ and the pattern they might create ‘inside’ a certain content of a statement.

As a result of this survey over metaphors, the difference between cases becomes clear, namely between the situation when the areas demarcated by concepts are rigidly fixed (because they are regarded as unalterable units) and the situation when the areas demarcated by concepts are alterable and can be combined in unpredictable ways. They are both means of explaining the characteristic of fruitfulness with images.

The correctness condition was metaphorically represented by the requirement that the decomposed whole (i.e. the initial whole, representing the left side of the identity) should be the same with the re-composed whole (such that an identity statement between them would be correct). The informativeness condition was metaphorically represented by the requirement that the decomposed whole should have constituent parts of a different shape than the recomposed whole, i.e., they should have a different structure.

4. INFORMATIVE IDENTITIES AND THE SOLUTION

I will compare in this section the previous passages, about fruitfulness, with passages where informativeness – or ‘greater cognitive value’ – is being discussed by Frege. In ‘Notes for Ludwig Darmstaedter,’ informativeness appears in its usual circumstance for Frege, namely as a property of an identity statement where the term on the left and on the right side of the identity have the same reference and different sense:

> The same object can be the meaning of different expressions, and anyone of them can have a sense different from any other. Identity of meaning can go hand in hand with difference of sense. This is what makes it possible for a sentence of the form '*A* = *B*' to express a thought with more content than one

> which merely exemplifies the law of identity. A statement in which something is recognized as the same again can be of far greater cognitive value than a particular case of the law of identity.[15]

Of course, the saga of the difference between *Sinn* and *Bedeutung* famously begins with 'Über Sinn und Bedeutung', where the difference in sense is again associated with richness in information:

> When we found 'a = a' and 'a = b' to have different cognitive values, the explanation is that for the purpose of knowledge, the sense of the sentence, viz., the thought expressed by it, is no less relevant than its meaning, i.e. its truth-value. If now a = b, then indeed what is meant be 'b' is the same as what is meant by 'a', and hence the truth value of 'a = b' is the same as that of 'a = a'. In spite of this, the sense of 'b' may differ from that of 'a', and thereby the thought expressed in 'a = b' differs from that of 'a = a'. In that case the two sentences do not have the same cognitive value.[16]

At the beginning of the article, in another famous example we meet the same view:

> Let a, b, c be the lines connecting the vertices of a triangle with the midpoints of the opposite sides. The point of intersection of a and b is then the same as the point of intersection of b and c. So we have different designations for the same point, and these names ('point of intersection of a and b', 'point of intersection of b and c') likewise indicate the mode of presentation; and hence the statement contains actual knowledge.[17]

It is clear that Frege speaks each time about identity statements (most of the times exemplifying them with equations) that increase knowledge when the senses of the expressions flanking the identity sign are different, but the reference is the same. In another clear passage from 'Logic in Mathematics' he affirms the same thing:

> We might say: if we designated the same number by '2+3' as by '5' then we should surely have to know that 5=2+3 straight off, and not need first to work it out. It is clearer if we take the case of larger numbers. It is surely not self-evident that 137+469=606; on the contrary we only come to see this as the result of first working it out. This sentence says much more than the sentence '606=606'; *the former increases our knowledge, not so the latter*.[18] So the thoughts contained in the two sentences must be different too. (…) One cannot fail to recognize that the thought expressed by '5=2+3' is different from that expressed by the sentece'5=5', although the difference only consists in the fact

[15] G. Frege, 'Notes for Ludwig Darmstaedter,' p. 255.
[16] G. Frege, 'On Sense and Meaning', in *Translations from Philosophical Writings,* p. 78.
[17] Ibidem, p.57.
[18] My italics.

> that in the second sentence ‘5’ , which designates the same number as ‘2+3’, takes the place of ‘2+3’. So the two signs are not equivalent from the point of view of the thought expressed, although they designate the very same number. Hence, I say that the signs ‘5’ and ‘2+3’ do indeed designate the same thing, but do not express the same *sense*. In the same way ‘Copernicus’ and ‘the author of the heliocentric view of the planetary system’ designate the same man, but have different senses; for the sentence ‘Copernicus is Copernicus’ and ‘Copernicus is the author of the heliocentric view of the planetary system’ do not express the same thought.[19]

The passages just quoted above show that for every *informative* identity given as an example there is present a specification of *difference in sense*. The problematic connection, however, from the present point of view, is not between difference in sense and informativeness, but between informativeness and fruitfulness. The advantages of considering them as different is that it would allow a coherent view of the status of definitions in Frege’s writings. On this interpretation, most definitions were seen as mere convenient abbreviations, with no particular informational value; however, some definitions had a special status, namely of ‘fruitful definitions’ (as Frege characterizes them above, in the quotes from *Grundlagen* and ‘Boole’s logical Calculus and the Concept-script’). The informational value of ‘fruitful definitions’, in this interpretation, was something in between the fully fledged ‘informativeness’ of Frege’s examples and the mere linguistic informativeness of useful abbreviations. Namely, it was the informational value brought by formal devices, devices that are useful in making logical connections and constructing inferential steps. I will try to defend a different view and, for this purpose, I will offer two arguments, one against this interpretation and another in favor of a different interpretation. It must be mentioned that the interpretation I am proposing here does not offer a solution for making Frege’s account of definitions coherent; this remains an unsolved problem.

My objection to the above interpretation of ‘fruitfulness’ as different from ‘informativeness’ is that Frege seems to attribute to his fruitful definitions more than the role ascribed by this interpretation. In other words, the point of disagreement between the interpretation described above and the one I propose is the informational value of Frege’s fruitful definitions. According to the previous interpretation, the value of Frege’s fruitful definitions is given by the fact that they are good heuristic means in achieving a formal proof, i.e., they are useful methods for arriving from one point ‘a’ to another point ‘n’ of a demonstration (– this is their ‘fruitfulness’ or utility). In this sense, fruitful definitions can be conceived in purely nominalistic terms (in a Hilbertian fashion, for example) or from a conceptualistic point of view, i.e., as providing a useful link between concepts. However, the manner in which fruitful definitions *cannot* be portrayed (when interpreted as mere heuristic, useful devices) is a *realistic* manner, i.e., it cannot be said that they provide information about how certain objects really are or what

[19] G. Frege, “Logic in Mathematics”, p. 225.

properties they have. My claim is that, no matter how much Frege can be (and he was) criticized or dismissed for it, he did have a realist view concerning his fruitful definitions. They were supposed to reveal real properties of mathematical objects. Noticeably, Frege attributes fruitfulness to his definition of number in *Grundlagen*. I think it is a plausible interpretation of Frege to say that when he defines the concept of number he wants to say what number really is.

Also, in support of this interpretation, can be brought Frege's affirmation (from the quote given above, paragraph 88 in *Grundlagen*) about the fact that Kant would call fruitful definitions 'synthetic' because of their greater cognitive value, in spite of the fact that they are provable by purely logical means, and therefore, are to be called, according to Frege, 'analytic': 'The conclusions we draw from it extend our knowledge, and ought therefore, on Kant's view, to be regarded as synthetic; and yet they can be proved by purely logical means, and are thus analytic.' This quote can be understood in the following manner: even though fruitful definitions are analytic (because provable by purely logical means), they could be called 'synthetic' by a Kantian because they bring veritable novelty, i.e., they genuinely increase knowledge (by contrast with the Kantian 'analytic judgments' which, famously, do not extend knowledge). In other words, Frege is saying that if we agree to distinguish between analytic and synthetic judgments according to their informational value, and therefore accept that 'analytic = not genuinely informative' and 'synthetic = genuinely informative', then the fruitful definitions should be called 'synthetic', i.e., genuinely informative.

The second argument brought here in favor of the thesis of the solidarity between informativeness and fruitfulness relies on finding a common trait, present in Frege's examples, for both informative identities and fruitful identities. Frege's examples for fruitful identities, from the passages quoted above are the definition of the continuity of a function, and the definition of a limit. On the other hand, the examples of informative identities, besides the famous Morning Star = Evening Star, are mainly mathematical examples of equations, like '137+469=606'. My claim is that in both cases, of examples of informative identities and of examples of fruitful definitions, there is one common and important element: on the right side of an informative identity as well as on the right side of a fruitful definition there is at least one concept that does not appear on the left side. This is, actually, what makes them informative, respectively fruitful, or, in Frege's words, this is why we feel that 'we are not simply taking out of the box again what we have just put into it' (i.e., we are not merely repeating the left side of the identity under a different form). In conclusion, this is, I believe, the common ground which supports the solidarity between 'fruitfulness' and 'informativeness'.

In conclusion, I believe that for Frege's fruitful definitions, it is more plausible that they have the role of genuine informative identities rather than of mere heuristic devices. Therefore, informativeness counts as main trait of fruitful definitions: they are 'fruitful' because they are informative.[20]

[20] The problem of the exact relation between 'fruitful' and 'informative', namely if all informative identities are fruitful or all fruitful indentities are informative or both , is left aside here. It is enough to establish that at least some fruitful identities are informative.

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